The Method of Moments in Electromagnetics

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Preface to the Free Edition

In 2026, I petitioned the publisher, Taylor & Francis Group / CRC Press, for a return of the copyright for this book, and they agreed. One of the primary reasons for this was the inability to make print corrections to known errors, update figures, and add small revisions and additions going forward. This is not the publisher's fault, as it's an inherent property of producing a printed work that is more or less fixed in time.

I have since decided to this book freely available. This free edition contains corrections for all known errors from the third edition, has significant updates to several chapters, and some new material. First, many of the MATLAB scripts that were used to generate data for the book have been updated, and made freely available for download. Next, Chapter 3 (Radiation and Scattering) has new sections that cover the Impedance Boundary Condition (IBC) as well as the resistive boundary (also known as resistive card). The chapters on thin wires and 2D problems have been updated significantly. The previous chapter on Physical Optics has been re-written into a more general treatment on high-frequency methods for solving the surface integral equations of scattering. Finally, the book has been typeset with a larger page format.

Note that going forward, existing material may continue to be modified, and new material may be added. A changelog will be made available that mentions any additions, changes, or error corrections.

Preface to the Third Edition

Shortly before the publication of the second edition of this book, I finished work on a new version of the *Serenity* RCS solver code. This new software added support for the Adaptive Cross Approximation (ACA), permitting a direct solution of the matrix system in compressed form. It also took advantage of the incredible computation power afforded by general purpose graphics processing units (GPUs), which can perform matrix operations much faster than traditional CPUs. Unfortunately, given the short timeframe, I was not able to add any material on the ACA to the second edition. Since that time, significant additional progress has been made on *Serenity*, as well as implementation of a Multi-Level Adaptive Cross Approximation (MLACA), which has recently begun to receive significant attention in the literature. Thus, the time now seems right to update the second edition with material on these very exciting, powerful algorithms.

This third edition adds a significant amount of new material. First, I have added a chapter that presents an in-depth treatment of the ACA as applied to the Method of Moments, as well as its software implementation on CPUs and GPUs. Next, I have added a chapter on the Multi-Level Adaptive Cross Approximation (MLACA), an advanced compression technique that uses a recursive, butterfly compression algorithm. This yields a far better level of compression than the regular ACA, at the expense of extra computation time. Finally, I have revisited and updated the existing material on the Fast Multipole Method (FMM), as well as the Multi-Level Fast Multipole Algorithm (MLFMA). This new treatment utilizes elements of the ACA approach to further reduce the the memory requirements of the FMM and MLFMA, and also presents compressed, block versions of the ILUT and SAI preconditioners. I have also done my best to correct known typographical errors from the second edition, and to incorporate feedback I've received from readers.

I began working on *Serenity* in 2000, and the first edition of this book in 2004. After 20 years of very hard work, I feel that it's finally time to take a rest. As a result, this will be the final edition of the book. Please send any errata to me via email at kalla@tripoint.org, and going forward I will maintain an up-to-date errata sheet for those interested.

Preface to the Second Edition

The first edition of this book was well received, and since its publication I received a lot of feedback and questions about its content. When I began considering a second edition, I knew that a key element had to be the treatment of dielectric materials. After I started working on the new material, it became apparent that to accomplish this goal I would have to re-write a significant amount of the existing material. The result is a book that I believe significantly improves and expands upon the first edition in several key areas:

1. The first edition focused exclusively on integral equations for conducting problems. However, dielectric materials are an important aspect of practical electromagnetic devices, and the ability to include these in computational codes is of increasing importance. In this second edition, we will derive coupled surface integral equations that will treat conducting as well as composite conducting/dielectric objects. This treatment extends to objects having multiple dielectric regions with interfaces and junctions. Though many sections of this book have been changed or updated to reflect new material, we will continue to use the approach in the first edition, where we progress gradually from simple to more complex problems and topics.

2. As my background is in the design and programming of software codes for radar cross section (RCS) prediction, the first edition focused almost exclusively on far-zone radiated fields and RCS. In this second edition, I have added more material involving the calculation of near fields.

3. Technology continues to march ever onward, and I have attempted to keep the material in this edition as up to date as possible. As a result, existing material has been updated, and material deemed to be outdated or no longer relevant has been removed.

4. Known errors from the first edition have been corrected, and concepts that were unclear or poorly explained have been updated for clarity. Many equations have also been reformatted, and figures adjusted or regenerated to fix errors or improve their appearance.

Preface

The Method of Moments (MoM) as applied to electromagnetic field problems was first described at length in Harrington's classic book[1], and was no doubt in use before that. Since then, computing technology has grown at a staggering pace, and the field of computational electromagnetics (CEM) has followed closely behind. Though many dissertations and journal papers have been dedicated to the MoM in electromagnetics, few textbooks have been written for those who are unfamiliar with it. For graduate students who are just beginning their study of CEM, or seasoned professionals solving real-world problems, this sets the barrier to entry fairly high.

Thus, this author felt that a new book describing the Moment Method as applied to electromagnetics was necessary. The first reason was because of the lack of a good introductory graduate-level text on the Moment Method. Though many universities offer courses in computational electromagnetics, the course material often comprises a disjoint collection of journal papers and copies of the instructor's personal notes. In the few books that do exist on the subject, the material is often presented at a high level with little to no attention paid to implementation details. Additionally, many of these often omit key information or refer to other papers (or worse, private communications), forcing the student to spend their time searching for missing information instead of focusing on course material. The second reason is the hope that a concise, up-to-date reference book will be of significant benefit to researchers and practicing professionals in the field of CEM. This book has several key features that set it apart from others of its kind:

1. *A straightforward, progressive introduction to field problems and the Moment Method.* This book begins by introducing the Method of Moments in the context of simple, electrostatic field problems. We then move to a review of frequency-domain electromagnetic theory, radiation equations, and the Green's functions for two- and three-dimensional objects. The surface equivalence theorem is then used to derive coupled surface integral equations of radiation and scattering for conducting and composite dielectric/conducting objects. Subsequent chapters are then dedicated to solving these integral equations for progressively more difficult problems involving thin wires, bodies of revolution, and two and three-dimensional bodies. With this material behind them, the student or researcher will be well-equipped for more advanced MoM topics encountered in the literature.

[1] R. F. Harrington *Field Computation by Moment Methods*, The Macmillan Company, 1968.

2. *A clear and concise summary of equations*. One of the fundamental problems encountered by this author is that clear expressions for the MoM matrix elements are almost never found in the literature. This book derives or summarizes the matrix elements used in every MoM problem, and all examples are computed using the expressions summarized in the text.

3. *A focus on radiation and scattering problems*. This book is primarily focused on scattering and radiation problems. Therefore, we will consider many practical examples such as antenna impedance calculation and radar cross section prediction, as well as calculation of near fields. Each example is presented in a straightforward manner with a careful explanation of the approach as well as explanation of the results.

4. *An up-to-date reference*. The material contained within is presented in the context of current-day computing technology, and includes up-to-date material on methods such as as the Fast Multipole Method and Adaptive Cross Approximation that are now commonly used in CEM.

5. *"Show your work."* In this work we attempt to avoid a so-called "it can be shown" approach, which can be detrimental to students, and instead focus on step-by-step derivations in key areas that are often glossed over or omitted in other texts.

This book is intended for a one- or two- semester course in computational electromagnetics and a reference for the practicing engineer. It is expected that the reader will be familiar with time-harmonic electromagnetic fields and vector calculus, as well as differential/integral equations and linear algebra. The reader should also have some basic experience with computer programming in a language such as C, C++ or FORTRAN, or a mathematical environment such as MATLAB. Because some of the expressions in this book require the calculation of special functions, the reader should be aware of what they are and know how to calculate them.

This book comprises nine chapters:

Chapter 1 presents a very brief overview of computational electromagnetics and some of the commonly used numerical techniques in this field. This will show the reader how surface integral equations and the Method of Moments fits into the world of CEM algorithms.

Chapter 2 introduces the Method of Moments in the context of simple electrostatic field problems. We then formalize the MoM, discuss point matching and Galerkin's Method, and present some commonly used two-dimensional basis functions.

Chapter 3 begins with Maxwell's Equations, and then discusses electromagnetic boundary conditions, formulations for radiation, vector potentials, Green's functions, and near and far fields. It then uses the equivalence principle to derive a set of coupled integral equations for scattering by multi-region, composite conducting/dielectric objects. These equations are then discretized in general form using the Method of Moments, and are refined and adapted to specific problems in later chapters.

Chapter 4 discusses the solution of matrix equations, including topics such as Gaussian Elimination, scalar and block-LU Decomposition, matrix condition numbers, and iterative solvers.

Chapter 5 considers radiation and scattering by thin wires. We derive the thin wire kernel and the Hallén and Pocklington thin wire integral equations, and show how these are solved. We then apply the Moment Method to thin wires of arbitrary shape, and then solve practical thin wire problems involving antennas and feedlines.

Chapter 6 applies the Moment Method to two-dimensional problems. The integral equation formulations in Chapter 3 are adapted to general, two-dimensional boundaries, and at TM and TE polarizations. The first half of the chapter considers conducting geometries, and the latter half then moves into dielectric and composite geometries and problems.

Chapter 7 considers three-dimensional objects which can be described as bodies of revolution (BoRs). Examples involving the radar cross section of conducting, dielectric, and composite objects are then presented. We will then discuss junctions between dielectric and conducting regions and their treatment, and consider some example problems having junctions.

Chapter 8 moves to three-dimensional (3D) surfaces of arbitrary shape. We discuss the modeling of surfaces using triangular facets, and devote significant effort to summarizing the expressions used to evaluate singular potential integrals over triangular elements. We will then present a method by which generalized junctions between regions can be treated in a systematic and straightforward manner. Next, we consider radar cross section problems involving conducting and composite objects, as well as some three-dimensional antenna problems.

Chapter 9 introduces the Adaptive Cross Approximation (ACA), which is a fast, kernel-agnostic algorithm that can be used to compress blocks of the MoM system matrix, as well as blocks in the corresponding LU-factored matrix. We will cover the ACA algorithm as well as software implementation details on regular CPUs as well as graphics processing units (GPUs). We will then consider problems of larger electrical size than were considered in Chapter 8.

Chapter 10 introduces the Multi-Level Adaptive Cross Approximation (MLACA), which comprises a hierarchical extension of the ACA presented in Chapter 9, allowing for much greater levels of compression at the expense of more computation. The MLACA algorithm and its implementation details on regular CPUs as well as GPUs is covered in detail. We will then consider problems of much larger electrical size than were considered in Chapter 9.

Chapter 11 discusses the Fast Multipole Method (FMM) and how it is used to accelerate the matrix-vector product in the iterative solution of 3D problems. We will cover the addition theorem, wave translation, and single and multi-level fast multipole algorithms. The treatment is concise and contains all the information required to successfully implement the FMM for conducting or composite geometries in new or existing moment method codes. We will then consider consider problems of even larger electrical size than were considered in Chapter 10.

Chapter 12 discusses high-frequency approaches to solving the surface integral equations of scattering. We cover Physical Optics (PO), decomposition of plane waves at dielectric interfaces, and present expressions for the PO scattered field from PEC and dielectric boundaries. We also briefly cover Shooting and Bouncing Rays (SBR), which combines Geometrical Optics (GO) and PO to treat multiple-bounce scattering.

Chapter 13 discusses some methods of numerical integration including the trape-

zoidal and Simpson's rule, area coordinates and Gaussian quadrature over planar triangular elements, as well as integrals of constant and linear sources over triangles.

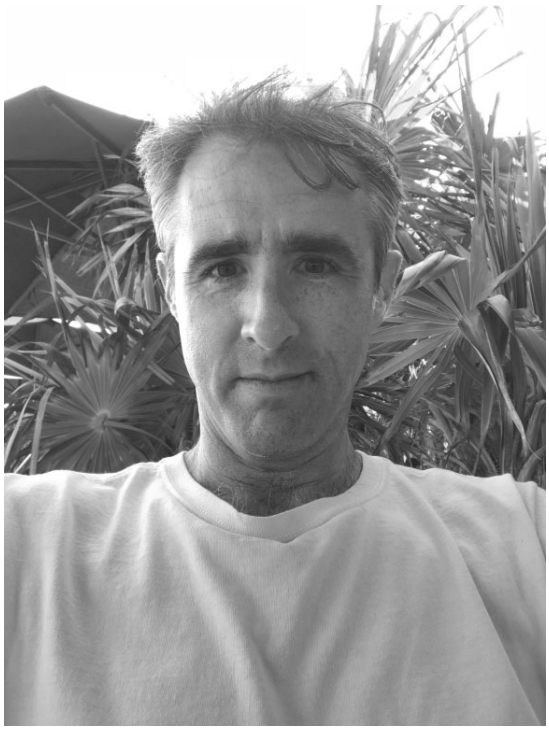
Throughout this text, an e^{jwt} time convention is assumed and suppressed throughout. We use SI units except in some examples where the test articles had dimensions in inches or feet. Numbers in parentheses ( ) refer to equations, and numbers in brackets [ ] are citations. Scalar quantities are written in italic font (a), vectors in bold lowercase font (**a**) with unit vectors denoted using the caret (**â**), and matrices in bold uppercase font (**A**).

Acknowledgments

This book came about because of my experience as a student in electromagnetics and as a developer of computational electromagnetics codes. My walk down this path undoubtedly began very early with my interest in the power of the computer, as well as the mysteries behind radio waves and antennas. I got my first taste as an undergraduate student at Auburn University, where my professor of electromagnetics, Dr. Thomas Shumpert, encouraged me to pursue the subject at the graduate level. It progressed further during my study at the University of Illinois, where I learned many great things about electromagnetics from Dr. Weng Chew and my graduate advisor Dr. Jianming Jin. During my subsequent work on the code suite known as *lucernhammer*, I encountered many hard problems in mathematics and software algorithms, programming, and optimization. Without the help of friends and colleagues, it is likely I would not have completed this path at all. Therefore, I would like to thank the following people who provided me with useful assistance, advice, and information when it was needed:

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