Chapter 3

Radiation and Scattering

In this chapter we will review Maxwell's equations and the electromagnetic boundary conditions, and derive the integral equations for radiation and scattering and the associated Green's functions needed to solve those equations. We will next derive the electric and magnetic vector potential and expressions for the radiated electric and magnetic near and far fields. We will then consider the solution of radiation problems by way of the equivalence principle, and derive coupled surface integral equations used to treat multi-region conducting, dielectric, and hybrid conducting/dielectric geometries. Finally, we will discretize these equations in a general sense using the Method of Moments, which will be applied to more specific two- and three-dimensional problems of increasing complexity in later chapters.

3.1 Maxwell's Equations

In a homogeneous dielectric region having constituent parameters ϵ and μ, the electric and magnetic fields must satisfy Maxwell's equations, which in the frequency domain are

$$\nabla \times \mathbf{E} = -\mathbf{M} - j\omega\mu\mathbf{H} \,, \tag{3.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E} \,, \tag{3.2}$$

$$\nabla \cdot \mathbf{D} = q_e \,, \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = q_m \,, \tag{3.4}$$

where $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, and the time dependence $e^{j\omega t}$ is assumed and suppressed throughout this text. Though the magnetic current $\mathbf{M}$ and charge q_m are not physically realizable quantities, they are useful mathematical tools in solving radiation and scattering problems, as we will show in Section 3.6.1.

3.2 Electromagnetic Boundary Conditions

At the interface between two regions R_1 and R_2, each having different dielectric parameters, the generalized electromagnetic boundary conditions can be written as

$$\hat{\mathbf{n}}_1 \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{M} \,, \tag{3.5}$$
$$\hat{\mathbf{n}}_1 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J} \,, \tag{3.6}$$
$$\hat{\mathbf{n}}_1 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = q_e \,, \tag{3.7}$$
$$\hat{\mathbf{n}}_1 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = q_m \,, \tag{3.8}$$

where any of $\mathbf{M}, \mathbf{J}, q_e$, or q_m may be present, and $\hat{\mathbf{n}}_1$ is the normal vector on the interface between R_1 and R_2 that points into R_1. At the interface between a dielectric (R_1) and a perfect electric conductor (PEC, R_2), the boundary conditions are

$$-\hat{\mathbf{n}}_1 \times \mathbf{E}_1 = 0 \,, \tag{3.9}$$
$$\hat{\mathbf{n}}_1 \times \mathbf{H}_1 = \mathbf{J} \,, \tag{3.10}$$
$$\hat{\mathbf{n}}_1 \cdot \mathbf{D}_1 = q_e \,, \tag{3.11}$$
$$\hat{\mathbf{n}}_1 \cdot \mathbf{B}_1 = 0 \,, \tag{3.12}$$

and between a dielectric (R_1) and perfect magnetic conductor (PMC, R_2) they are

$$-\hat{\mathbf{n}}_1 \times \mathbf{E}_1 = \mathbf{M} \,, \tag{3.13}$$
$$\hat{\mathbf{n}}_1 \times \mathbf{H}_1 = 0 \,, \tag{3.14}$$
$$\hat{\mathbf{n}}_1 \cdot \mathbf{D}_1 = 0 \,, \tag{3.15}$$
$$\hat{\mathbf{n}}_1 \cdot \mathbf{B}_1 = q_m \,. \tag{3.16}$$

In this book we will consider dielectric and PEC regions only.

3.3 Formulations for Radiation

The electromagnetic radiation problem involves the calculation of fields everywhere in space due to a set of electric and magnetic currents. Scattering problems can be treated as radiation problems where these currents are generated by other currents or impressed fields. Let us start from first principles and derive the expression for the electric field in the presence of electric and magnetic currents **J** and **M**. To begin, we take the curl of (3.1) and combine with (3.2) to get

$$\nabla \times \nabla \times \mathbf{E} = -j\omega\mu\nabla \times \mathbf{H} - j\omega\mu\mathbf{J} - \nabla \times \mathbf{M} = \omega^2\mu\epsilon\mathbf{E} - j\omega\mu\mathbf{J} - \nabla \times \mathbf{M} \tag{3.17}$$

or

$$\nabla \times \nabla \times \mathbf{E} - \omega^2\mu\epsilon\mathbf{E} = -j\omega\mu\mathbf{J} - \nabla \times \mathbf{M} \,. \tag{3.18}$$

Using the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \,, \tag{3.19}$$

we can write (3.18) as

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} - k^2\mathbf{E} = -j\omega\mu\mathbf{J} - \nabla \times \mathbf{M}\,, \tag{3.20}$$

where the wavenumber $k = \omega\sqrt{\mu\epsilon} = 2\pi/\lambda$. Substituting (3.3) into the above we get

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = j\omega\mu\mathbf{J} + \frac{\nabla q_e}{\epsilon} - \nabla \times \mathbf{M}\,. \tag{3.21}$$

Employing the equation of continuity

$$\nabla \cdot \mathbf{J} = -j\omega q_e \tag{3.22}$$

allows us to then obtain

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = j\omega\mu\mathbf{J} - \frac{1}{j\omega\epsilon}\nabla(\nabla \cdot \mathbf{J}) - \nabla \times \mathbf{M}\,. \tag{3.23}$$

Since Maxwell's equations are linear, we can consider **J** and **M** to be a superposition of point sources distributed over some volume. Therefore, if we know the response of a point source, we can solve the original problem by integrating that response over the volume. We now make use of this idea to convert (3.23) into an integral equation. To do so, we introduce the *Green's function* $G(\mathbf{r}, \mathbf{r}')$, which satisfies the scalar Helmholtz equation [1]:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + k^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r}, \mathbf{r}')\,. \tag{3.24}$$

Assuming that we know $G(\mathbf{r}, \mathbf{r}')$, the radiated electric field is then

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int_V G(\mathbf{r}, \mathbf{r}') \left[1 + \frac{1}{k^2}\nabla'\nabla' \cdot \right]\mathbf{J}(\mathbf{r}')\, d\mathbf{r}' - \int_V G(\mathbf{r}, \mathbf{r}')\nabla' \times \mathbf{M}(\mathbf{r}')\, d\mathbf{r}'\,, \tag{3.25}$$

and by a similar derivation, the radiated magnetic field is

$$\mathbf{H}(\mathbf{r}) = -j\omega\epsilon \int_V G(\mathbf{r}, \mathbf{r}') \left[1 + \frac{1}{k^2}\nabla'\nabla' \cdot \right]\mathbf{M}(\mathbf{r}')\, d\mathbf{r}' + \int_V G(\mathbf{r}, \mathbf{r}')\nabla' \times \mathbf{J}(\mathbf{r}')\, d\mathbf{r}'\,. \tag{3.26}$$

We will next solve (3.24) for $G(\mathbf{r}, \mathbf{r}')$ in two and three dimensions. To do so, we will first consider the homogeneous version, and then match the boundary conditions of the inhomogeneous case to obtain a unique solution.

3.3.1 Three-Dimensional Green's Function

As $G(\mathbf{r}, \mathbf{r}')$ is the solution for a point electromagnetic source, it must exhibit spherical symmetry in three dimensions. Therefore, we will retain only the radial term of the Laplacian and write

$$\nabla^2 G = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dG}{dr}) = \frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr}\,. \tag{3.27}$$

Next, we observe that

$$\frac{1}{r}\frac{d^2(rG)}{dr^2} = \frac{1}{r}\frac{d}{dr}\Big[r\frac{dG}{dr} + G\Big] = \frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} \tag{3.28}$$

and if we substitute in the function rG for $r > 0$, we can write

$$\frac{d^2(rG)}{dr^2} + k^2(rG) = 0 \,. \tag{3.29}$$

The solution to the above is

$$G(r) = A\frac{e^{-jkr}}{r} + B\frac{e^{jkr}}{r} \,, \tag{3.30}$$

where $r = |\mathbf{r} - \mathbf{r}'|$. Noting that the solution must contain only outgoing waves, we retain only the first term in (3.30), yielding

$$G(r) = A\frac{e^{-jkr}}{r} \,. \tag{3.31}$$

It should be pointed out that the phase convention of the exponential in the Green's function is not standard throughout the literature. The phase as defined in (3.31) is standard in most electrical engineering references; however, other references [2, 3] define the Green's function with a positive exponent, assuming the time harmonic term $e^{-j\omega t}$. Similarly, the two-dimensional Green's function in (3.49) would instead use the Hankel function of the first kind.

We must now match the boundary conditions to determine a unique solution. First, the amplitude of the outgoing wave must decay to zero with increasing r, requiring that $G(r) \to 0$ as $r \to \infty$. By inspection, we see that this is already satisfied. Next, we must match the Green's function at the point source ($r = 0$), which will allow us to determine A. To do so, we integrate (3.24) over a spherical volume of radius a around the source. Substituting (3.31) for $G(r)$, we get

$$A\int_V \left[\nabla \cdot \nabla\Big(\frac{e^{-jkr}}{r}\Big) + k^2\frac{e^{-jkr}}{r}\right] dV = -1 \,, \tag{3.32}$$

and to evaluate the first term, we use the divergence theorem to write

$$\int_V \nabla \cdot \nabla\Big(\frac{e^{-jkr}}{r}\Big) dV = \int_S \hat{\mathbf{n}} \cdot \nabla\Big(\frac{e^{-jkr}}{r}\Big) dS \,. \tag{3.33}$$

On the sphere, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, therefore

$$\int_S \hat{\mathbf{r}} \cdot \nabla(\frac{e^{-jkr}}{r})\, dS = \int_S \frac{\partial}{\partial r}(\frac{e^{-jkr}}{r})\, dS \tag{3.34}$$

which is

$$4\pi a^2 \left[\frac{\partial}{\partial r}\Big(\frac{e^{-jkr}}{r}\Big)\right]_{r=a} . \tag{3.35}$$

Taking the limit as $a \to 0$ we get

$$\lim_{a\to 0} 4\pi a^2 \left[\frac{\partial}{\partial r}\Big(\frac{e^{-jkr}}{r}\Big)\right]_{r=a} = -4\pi \,. \tag{3.36}$$

The second term is

$$k^2 \int_0^a \frac{e^{-jkr}}{r} 4\pi r^2 dr = 4\pi k^2 \int_0^a r e^{-jkr} dr \; . \tag{3.37}$$

By inspection, this integral tends to zero as $a \to 0$. Therefore,

$$A = \frac{1}{4\pi} \tag{3.38}$$

and finally,

$$G(r) = \frac{e^{-jkr}}{4\pi r} \; . \tag{3.39}$$

which is the electrodynamic Green's function in three dimensions.

3.3.2 Two-Dimensional Green's Function

Two-dimensional problems are those where the currents, surfaces, and fields extend to $\pm\infty$ along the $\hat{\mathbf{z}}$ axis. As such, there is no wave propagation in the $\hat{\mathbf{z}}$ direction. For these problems we will work in the cylindrical coordinate system (ρ, ϕ, z) where $\boldsymbol{\rho} = \rho\cos\phi\ \hat{\mathbf{x}} + \rho\sin\phi\ \hat{\mathbf{y}}$, and surface integrals are performed in the (x, y) plane $(z = 0)$. In two dimensions, (3.24) can be written as

$$\nabla^2 G(\boldsymbol{\rho}, \boldsymbol{\rho}') + k^2 G(\boldsymbol{\rho}, \boldsymbol{\rho}') = -\delta(\boldsymbol{\rho}, \boldsymbol{\rho}') \; . \tag{3.40}$$

The solutions to the above for the homogeneous case are the Hankel functions of the first and second kinds of order zero. Since we know the solution contains only outgoing waves, we can then write

$$G(\rho) = A H_0^{(2)}(k\rho) \tag{3.41}$$

where $\rho = |\boldsymbol{\rho} - \boldsymbol{\rho}'|$. To obtain A, we use the small-argument approximation to the Hankel function [4]

$$H_0^{(2)}(k\rho) \approx 1 - j\frac{2}{\pi}\log\big(\frac{\gamma k\rho}{2}\big) \qquad \rho \to 0 \; , \tag{3.42}$$

and integrate (3.40) over a very small circle of radius a centered at the origin, yielding

$$A \int_S \Big[\nabla^2 + k^2\Big]\Big[1 - j\frac{2}{\pi}\log\big(\frac{\gamma k\rho}{2}\big)\Big]\, dS = -1 \; . \tag{3.43}$$

Noting that $\nabla^2 f = \nabla \cdot \big(\nabla f\big)$, the first term can be converted to a line integral using the divergence theorem, yielding

$$-j\frac{2}{\pi}\int_0^{2\pi} \nabla\Big[\log\big(\frac{\gamma k\rho}{2}\big)\Big]\, \rho\, d\phi = -4j \; . \tag{3.44}$$

The second term is

$$k^2 \int_0^a \Big[1 - j\frac{2}{\pi}\log\big(\frac{\gamma k\rho}{2}\big)\Big] 2\pi\rho\, d\rho \; . \tag{3.45}$$

The first part of (3.45) goes to zero as $a \to 0$. Integrating the second part yields

$$4jk^2 \int_0^a \log\Big(\frac{\gamma k\rho}{2}\Big)\rho\, d\rho = \left[\frac{\rho^2}{2}\log\Big(\frac{\gamma k\rho}{2}\Big) - \frac{\rho^2}{4}\right]\Bigg|_0^a . \tag{3.46}$$

The above goes to zero, since

$$\lim_{\rho\to 0} \rho^2 \log\rho = 0 . \tag{3.47}$$

Therefore,

$$A = -\frac{j}{4} \tag{3.48}$$

and finally,

$$G(\rho) = -\frac{j}{4}H_0^{(2)}(k\rho) , \tag{3.49}$$

which is the electrodynamic Green's function in two dimensions.

3.4 Vector Potentials

In Section 3.3, we derived expressions that allow us to determine the radiated field everywhere in space from electric and magnetic current distributions. In this section we will derive alternative expressions for the radiated fields in terms of vector potentials, also commonly used in radiation and scattering problems. We will then address the differences between the formulations in the following section.

3.4.1 Magnetic Vector Potential

We will first derive the magnetic vector potential for a homogeneous, source-free region. We begin by observing that since the magnetic field $\mathbf{H}$ is always solenoidal, it can be written as the curl of another vector $\mathbf{A}$, which is

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} . \tag{3.50}$$

Substituting the above into (3.1), we get

$$\nabla\times\mathbf{E} = -j\omega\nabla\times\mathbf{A} , \tag{3.51}$$

which can be written as

$$\nabla\times(\mathbf{E} + j\omega\mathbf{A}) = 0 . \tag{3.52}$$

We next use the identity

$$\nabla\times(-\nabla\Phi_e) = 0 \tag{3.53}$$

to write

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi_e , \tag{3.54}$$

where Φ_e is an arbitrary electric *scalar potential*. We next use the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} \,, \tag{3.55}$$

and take the curl of both sides of (3.50) allowing us to write

$$\mu\nabla \times \mathbf{H} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} \,, \tag{3.56}$$

and combining this with (3.2) leads to

$$\mu\mathbf{J} + j\omega\mu\epsilon\mathbf{E} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} \,. \tag{3.57}$$

Substituting (3.54) into the above leads to

$$\mu\mathbf{J} + j\omega\mu\epsilon(-j\omega\mathbf{A} - \nabla\Phi_e) = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} \,, \tag{3.58}$$

which is

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\mu\epsilon\Phi_e) \,. \tag{3.59}$$

We have already defined the curl of $\mathbf{A}$ in (3.50), however we have not yet defined its divergence. We are therefore free to make it whatever we wish as long as we remain consistent with that definition elsewhere. Therefore, if we make the following assignment

$$\Phi_e = -\frac{1}{j\omega\mu\epsilon}\nabla \cdot \mathbf{A} \,, \tag{3.60}$$

(the Lorentz gauge) we can simplify (3.59) to yield

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J} \,, \tag{3.61}$$

which is an inhomogeneous vector Helmholtz equation for $\mathbf{A}$. Using the Green's function, $\mathbf{A}(\mathbf{r})$ can be written as

$$\mathbf{A}(\mathbf{r}) = \mu \int_V G(\mathbf{r}, \mathbf{r}')\, \mathbf{J}(\mathbf{r}')\, d\mathbf{r}' \,, \tag{3.62}$$

and the corresponding electric field is then

$$\mathbf{E}(\mathbf{r}) = -j\omega\Big[1 + \frac{1}{k^2}\nabla\nabla \cdot \Big]\mathbf{A}(\mathbf{r}) \,. \tag{3.63}$$

3.4.1.1 Three-Dimensional Magnetic Vector Potential

Substitution of (3.39) into (3.62) yields the magnetic vector potential in three dimensions, which is

$$\mathbf{A}(\mathbf{r}) = \mu \int_V \mathbf{J}(\mathbf{r}')\, \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}\, d\mathbf{r}' \,. \tag{3.64}$$

3.4.1.2 Two-Dimensional Magnetic Vector Potential

The corresponding magnetic vector potential in two dimensions is obtained via (3.49), and is

$$\mathbf{A}(\boldsymbol{\rho}) = -j\frac{\mu}{4} \int_S \mathbf{J}(\boldsymbol{\rho}')H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)\, d\boldsymbol{\rho}' \,. \tag{3.65}$$

3.4.2 Electric Vector Potential

Using the symmetry of Maxwell's equations, we can derive similar expressions for the electric vector potential $\mathbf{F}$. We will not repeat the derivation, and simply summarize these expressions below:

$$\mathbf{E} = -\frac{1}{\epsilon}\nabla \times \mathbf{F} \,, \tag{3.66}$$

$$\Phi_m = -\frac{1}{j\omega\mu\epsilon}\nabla \cdot \mathbf{F} \,, \tag{3.67}$$

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\epsilon\mathbf{M} \,, \tag{3.68}$$

$$\mathbf{H}(\mathbf{r}) = -j\omega\Big[1 + \frac{1}{k^2}\nabla\nabla\cdot\Big]\mathbf{F}(\mathbf{r}) \,, \tag{3.69}$$

$$\mathbf{F}(\mathbf{r}) = \epsilon\int_V \mathbf{M}(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.70}$$

3.4.2.1 Three-Dimensional Electric Vector Potential

Substitution of (3.39) into (3.70) yields the electric vector potential in three dimensions, which is

$$\mathbf{F}(\mathbf{r}) = \epsilon\int_V \mathbf{M}(\mathbf{r}')\, \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d\mathbf{r}' \,. \tag{3.71}$$

3.4.2.2 Two-Dimensional Electric Vector Potential

Similarly, the two-dimensional electric vector potential is obtained via (3.49), and is

$$\mathbf{F}(\boldsymbol{\rho}) = -j\frac{\epsilon}{4}\int_S \mathbf{M}(\boldsymbol{\rho}')H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\, d\boldsymbol{\rho}' \,. \tag{3.72}$$

3.4.3 Total Fields

Using the results from Sections 3.4.1 and 3.4.2, we can write the total electric field $\mathbf{E}(\mathbf{r})$ as

$$\mathbf{E}(\mathbf{r}) = -j\omega\Big[\mathbf{A}(\mathbf{r}) + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{A}(\mathbf{r})\Big] - \frac{1}{\epsilon}\nabla\times\mathbf{F}(\mathbf{r}) \,, \tag{3.73}$$

and the magnetic field $\mathbf{H}(\mathbf{r})$ as

$$\mathbf{H}(\mathbf{r}) = -j\omega\Big[\mathbf{F}(\mathbf{r}) + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{F}(\mathbf{r})\Big] + \frac{1}{\mu}\nabla\times\mathbf{A}(\mathbf{r}) \,. \tag{3.74}$$

In *operator notation*, we can rewrite (3.73) and (3.74) as

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu(\mathcal{L}\mathbf{J})(\mathbf{r}) - (\mathcal{K}\mathbf{M})(\mathbf{r}) \tag{3.75}$$

and

$$\mathbf{H}(\mathbf{r}) = -j\omega\epsilon(\mathcal{L}\mathbf{M})(\mathbf{r}) + (\mathcal{K}\mathbf{J})(\mathbf{r}) \,, \tag{3.76}$$

where the operators $\mathcal{L}$ and $\mathcal{K}$ are

$$(\mathcal{L}\mathbf{X})(\mathbf{r}) = \Big[1 + \frac{1}{k^2}\nabla\nabla\cdot\Big] \int_V G(\mathbf{r},\mathbf{r}')\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \tag{3.77}$$

and

$$(\mathcal{K}\mathbf{X})(\mathbf{r}) = \nabla \times \int_V G(\mathbf{r},\mathbf{r}')\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.78}$$

3.4.4 Comparison of Radiation Formulas

The expressions for the electric and magnetic fields in (3.73) and (3.74) are almost identical to those of (3.25) and (3.26) except that the differential operators are distributed differently. It is important that we show their equivalence, as the reader will likely encounter both forms in the literature, as well as expressions where the operators are distributed differently than shown in this book. We will also use techniques from this section to redistribute differential operators in cases where it simplifies evaluation of the MoM matrix elements.

We now show that the differential operators may be moved under the integral sign (or onto the Green's function) provided the sources are of finite extent. This equivalence is used repeatedly when we later evaluate MoM matrix elements. To demonstrate this equivalence, we must show that

$$\mathbf{I}_1(\mathbf{r}) = \nabla\nabla\cdot\int_V G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' = \int_V G(\mathbf{r},\mathbf{r}')\nabla'\nabla'\cdot\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \tag{3.79}$$

and

$$\mathbf{I}_2(\mathbf{r}) = \nabla\times\int_V G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' = \int_V G(\mathbf{r},\mathbf{r}')\nabla'\times\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.80}$$

Let us begin with $\mathbf{I}_1(\mathbf{r})$. Using the vector identity

$$\nabla\cdot\big[G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\big] = \big[\nabla G(\mathbf{r},\mathbf{r}')\big]\cdot\mathbf{X}(\mathbf{r}') + G(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{X}(\mathbf{r}') \tag{3.81}$$

and the fact that $\mathbf{X}(\mathbf{r}')$ is not a function of the unprimed coordinates, we can write

$$\mathbf{I}_1(\mathbf{r}) = \nabla\nabla\cdot\int_V G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' = \nabla\int_V \big[\nabla G(\mathbf{r},\mathbf{r}')\big]\cdot\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.82}$$

Using the symmetry of the Green's function,

$$\nabla G(\mathbf{r},\mathbf{r}') = -\nabla' G(\mathbf{r},\mathbf{r}') \,, \tag{3.83}$$

the above becomes

$$\mathbf{I}_1(\mathbf{r}) = -\nabla\int_V \big[\nabla' G(\mathbf{r},\mathbf{r}')\big]\,\mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.84}$$

Using the previous vector identity we can write the above as a sum of two integrals as

$$\mathbf{I}_1(\mathbf{r}) = \nabla \int_V G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' - \nabla \int_V \nabla' \cdot \left[G(\mathbf{r},\mathbf{r}')\, \mathbf{X}(\mathbf{r}')\right] d\mathbf{r}' \,. \tag{3.85}$$

We then make use of the divergence theorem to write the second term as

$$\mathbf{I}_1(\mathbf{r}) = \nabla \int_V G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' - \nabla \oint_S \hat{\mathbf{n}} \cdot \left[G(\mathbf{r},\mathbf{r}')\, \mathbf{X}(\mathbf{r}')\right] d\mathbf{r}' \,. \tag{3.86}$$

Assuming $\mathbf{X}(\mathbf{r}')$ is of finite extent and completely inside V, it will be zero on S. Thus, the second integral is zero and we are left with

$$\mathbf{I}_1(\mathbf{r}) = \nabla \int_V G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.87}$$

We must now move the remaining gradient operator under the integral sign, and doing so results in

$$\mathbf{I}_1(\mathbf{r}) = -\int_V \nabla' G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,, \tag{3.88}$$

where we have again employed the symmetry of the Green's function. We must now move the gradient so that it operates on the $\nabla' \cdot \mathbf{X}(\mathbf{r}')$ term. To do so, we use the vector identity

$$\nabla' G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}') = \nabla'\left[G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}')\right] - G(\mathbf{r},\mathbf{r}')\nabla'\nabla' \cdot \mathbf{X}(\mathbf{r}') \tag{3.89}$$

to write

$$\mathbf{I}_1(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')\nabla'\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' - \int_V \nabla'\left[G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}')\right] d\mathbf{r}' \,. \tag{3.90}$$

To show that the second term is zero, we apply the following result of the divergence theorem

$$\int_V \nabla f(\mathbf{r})\, dV = \oint_S f(\mathbf{r})\, d\mathbf{S} \tag{3.91}$$

to write the above as

$$\mathbf{I}_1(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')\nabla'\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' - \oint_S G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.92}$$

Again making V large enough such that $\mathbf{X}(\mathbf{r}')$ has zero value on S, we are left with

$$\mathbf{I}_1(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')\nabla'\nabla' \cdot \mathbf{X}(\mathbf{r}')\, d\mathbf{r}' \,, \tag{3.93}$$

which is the expected result. We next focus our attention on $\mathbf{I}_2(\mathbf{r})$. Using the vector identity

$$\nabla \times \left[G(\mathbf{r},\mathbf{r}')\, \mathbf{X}(\mathbf{r}')\right] = G(\mathbf{r},\mathbf{r}')\nabla \times \mathbf{X}(\mathbf{r}') + \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{X}(\mathbf{r}') \,, \tag{3.94}$$

we can write

$$\nabla \times \int_V G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\,d\mathbf{r}' = \int_V G(\mathbf{r},\mathbf{r}')\nabla \times \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' + \int_V \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' \,. \tag{3.95}$$

Noting that $\nabla G(\mathbf{r},\mathbf{r}') = -\nabla' G(\mathbf{r},\mathbf{r}')$ and that $\nabla \times \mathbf{X}(\mathbf{r}') = 0$ (as $\mathbf{X}$ does not depend on the unprimed coordinates), we write the above as

$$\nabla \times \int_V G(\mathbf{r},\mathbf{r}')\,\mathbf{X}\,d\mathbf{r}' = -\int_V \nabla' G(\mathbf{r},\mathbf{r}') \times \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' \,. \tag{3.96}$$

Again using the identity (3.94) but on primed coordinates, we write

$$\begin{aligned} -\int_V \nabla' G(\mathbf{r},\mathbf{r}') \times\ \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' &= \int_V G(\mathbf{r},\mathbf{r}')\nabla' \times\ \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' \\ &\quad - \int_V \nabla' \times \left[G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\right] d\mathbf{r}' \,, \end{aligned} \tag{3.97}$$

and using the following result of the divergence theorem

$$\int_V \nabla \times \mathbf{X}(\mathbf{r})\,dV = \oint_S \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{X}(\mathbf{r})\,dS \,, \tag{3.98}$$

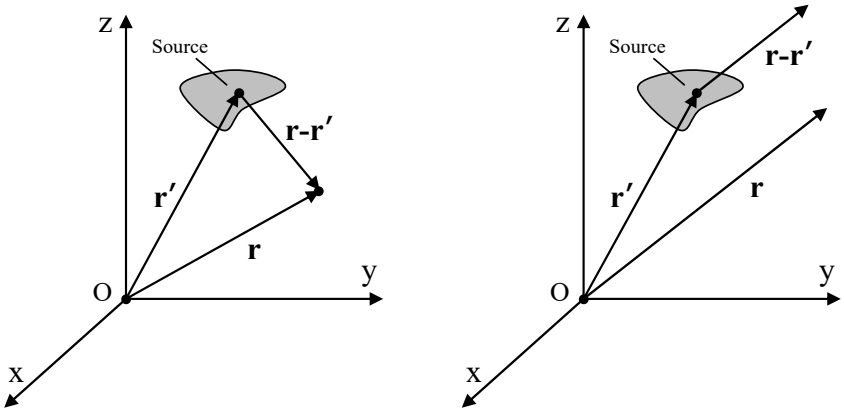
we can write (3.4.4) as

$$\begin{aligned} -\int_V \nabla' G(\mathbf{r},\mathbf{r}') \times\ \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' &= \int_V G(\mathbf{r},\mathbf{r}')\nabla' \times\ \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' \\ &\quad - \oint_S \hat{\mathbf{n}} \times \left[G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\right] d\mathbf{r}' \,. \end{aligned} \tag{3.99}$$

Making V large enough such that $\mathbf{X}(\mathbf{r}')$ has zero value on S, the second term on the right hand side of (3.4.4) goes to zero and we are left with

$$\nabla \times \int_V G(\mathbf{r},\mathbf{r}')\,\mathbf{X}\,d\mathbf{r}' = \int_V G(\mathbf{r},\mathbf{r}')\nabla' \times \mathbf{X}(\mathbf{r}')\,d\mathbf{r}' \,, \tag{3.100}$$

which is again the expected result.



(a) Near Field Geometry (b) Far Field Geometry

Figure 3.1: Near and Far Field Geometry

3.5 Near and Far Field

The radiation equations presented thus far are general formulas that can be used to compute the electric and magnetic fields anywhere. Usually though, one is interested in the fields in one of two regions: the *near field* region or the *far field* region. The near field region is located close to the source, where the fields comprise a superposition of stored (reactive) as well as radiated power. The near field region is of interest in applications such as antenna design, electromagnetic interference (EMI), and packaging of electronic components. The far field region is that which is far away from the source, where the fields have transitioned into purely radiative plane waves. In this section we will derive expressions for the near and far fields in two and three dimensions.

3.5.1 Three-Dimensional Near Field

Let us obtain expressions for the three-dimensional near fields at a point close to a source, as illustrated in Figure 3.1a. Recall that the magnetic field radiated by an electric current can be obtained from (3.50) and (3.62) and is

$$\mathbf{H}_A(\mathbf{r}) = \frac{1}{\mu}\nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \int_V \mathbf{J}(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.101}$$

Moving the curl operator under the integral sign and using the vector identity (3.94) and the fact that $\nabla \times \mathbf{J}(\mathbf{r}') = 0$, we then write

$$\mathbf{H}_A(\mathbf{r}) = \int_V \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{J}(\mathbf{r}')\, d\mathbf{r}' \,. \tag{3.102}$$

Taking the gradient of the 3D Green's function (3.39) yields

$$\nabla\left(\frac{e^{-jkr}}{4\pi r}\right) = -(\mathbf{r}-\mathbf{r}')\left(\frac{1+jkr}{4\pi r^3}\right)e^{-jkr} \,, \tag{3.103}$$

and using this result we can then write the magnetic field as

$$\mathbf{H}_A(\mathbf{r}) = -\frac{1}{4\pi}\int_V \left[(\mathbf{r}-\mathbf{r}')\times\mathbf{J}(\mathbf{r}')\right]\frac{1+jkr}{r^3}e^{-jkr}\,d\mathbf{r}' \,. \tag{3.104}$$

Expanding into Cartesian components, the above can be written as

$$\begin{aligned}
H_{A,x}(\mathbf{r}) &= \frac{1}{4\pi}\int_V \Big[\Delta_z J_y - \Delta_y J_z\Big]\frac{1+jkr}{r^3}e^{-jkr}\,d\mathbf{r}' \,,\\
H_{A,y}(\mathbf{r}) &= \frac{1}{4\pi}\int_V \Big[\Delta_x J_z - \Delta_z J_x\Big]\frac{1+jkr}{r^3}e^{-jkr}\,d\mathbf{r}' \,,\\
H_{A,z}(\mathbf{r}) &= \frac{1}{4\pi}\int_V \Big[\Delta_y J_x - \Delta_x J_y\Big]\frac{1+jkr}{r^3}e^{-jkr}\,d\mathbf{r}' \,,
\end{aligned} \tag{3.105}$$

where

$$\Delta_x = x - x' \,, \qquad \Delta_y = y - y' \,, \qquad \Delta_z = z - z' \,, \tag{3.106}$$

and

$$r = |\mathbf{r}-\mathbf{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \,. \tag{3.107}$$

Using (3.2), the corresponding electric field is

$$\begin{aligned}
E_{A,x}(\mathbf{r}) &= -\frac{j}{4\pi\omega\epsilon}\int_V \Big(2C_1 J_x + C_2\big[\Delta_x(\Delta_z J_z + \Delta_y J_y) - J_x(\Delta_y^2+\Delta_z^2)\big]\Big)\,d\mathbf{r}' \,,\\
E_{A,y}(\mathbf{r}) &= -\frac{j}{4\pi\omega\epsilon}\int_V \Big(2C_1 J_y + C_2\big[\Delta_y(\Delta_x J_x + \Delta_z J_z) - J_y(\Delta_z^2+\Delta_x^2)\big]\Big)\,d\mathbf{r}' \,,\\
E_{A,z}(\mathbf{r}) &= -\frac{j}{4\pi\omega\epsilon}\int_V \Big(2C_1 J_z + C_2\big[\Delta_z(\Delta_y J_y + \Delta_x J_x) - J_z(\Delta_x^2+\Delta_y^2)\big]\Big)\,d\mathbf{r}' \,,
\end{aligned} \tag{3.108}$$

where

$$C_1 = \frac{1+jkr}{r^3}\,e^{-jkr} \tag{3.109}$$

and

$$C_2 = \frac{3+3jkr-k^2r^2}{r^5}\,e^{-jkr} \,. \tag{3.110}$$

Similarly, the electric field radiated by a magnetic current is

$$\mathbf{E}_F(\mathbf{r}) = -\frac{1}{\epsilon}\nabla\times\mathbf{F}(\mathbf{r}) = -\nabla\times\int_V \mathbf{M}(\mathbf{r}')\,G(\mathbf{r},\mathbf{r}')\,d\mathbf{r}' \tag{3.111}$$

and its rectangular components are

$$\begin{aligned}
E_{F,x}(\mathbf{r}) &= -\frac{1}{4\pi}\int_V \Big[\Delta_z M_y - \Delta_y M_z\Big]\frac{1+jkr}{r^3}e^{-jkr}\,d\mathbf{r}' \,,\\
E_{F,y}(\mathbf{r}) &= -\frac{1}{4\pi}\int_V \Big[\Delta_x M_z - \Delta_z M_x\Big]\frac{1+jkr}{r^3}e^{-jkr}\,d\mathbf{r}' \,,\\
E_{F,z}(\mathbf{r}) &= -\frac{1}{4\pi}\int_V \Big[\Delta_y M_x - \Delta_x M_y\Big]\frac{1+jkr}{r^3}e^{-jkr}\,d\mathbf{r}' \,,
\end{aligned} \tag{3.112}$$

and using (3.1), the corresponding magnetic field is

$$\begin{aligned}
H_{F,x}(\mathbf{r}) &= -\frac{j}{4\pi\omega\mu}\int_V \Big(2C_1 M_x + C_2\big[\Delta_x(\Delta_z M_z + \Delta_y M_y) - M_x(\Delta_y^2 + \Delta_z^2)\big]\Big)\, d\mathbf{r}' \,, \\
H_{F,y}(\mathbf{r}) &= -\frac{j}{4\pi\omega\mu}\int_V \Big(2C_1 M_y + C_2\big[\Delta_y(\Delta_x M_x + \Delta_z M_z) - M_y(\Delta_z^2 + \Delta_x^2)\big]\Big)\, d\mathbf{r}' \,, \\
H_{F,z}(\mathbf{r}) &= -\frac{j}{4\pi\omega\mu}\int_V \Big(2C_1 M_z + C_2\big[\Delta_z(\Delta_y M_y + \Delta_x M_x) - M_z(\Delta_x^2 + \Delta_y^2)\big]\Big)\, d\mathbf{r}' \,.
\end{aligned} \tag{3.113}$$

Note that terms C_1 and C_2 in these equations become singular for field points located very close to the source, and numerical integration algorithms alone may yield inaccurate results. In such cases, a more advanced singularity treatment may be needed, such as those outlined in [5, 6] for current distributions on planar triangles. Some of these techniques, such as singularity extraction or Duffy transform, are discussed in detail in later chapters.

3.5.2 Two-Dimensional Near Field

Following (3.101) and using the two-dimensional Green's function (3.49), the magnetic field $\mathbf{H}(\boldsymbol{\rho})$ can be written as

$$\mathbf{H}_A(\boldsymbol{\rho}) = \frac{1}{\mu}\nabla \times \mathbf{A}(\boldsymbol{\rho}) = -\frac{j}{4}\int_S \nabla H_0^{(2)}(k\rho) \times \mathbf{J}(\boldsymbol{\rho}')\, d\boldsymbol{\rho}' \,. \tag{3.114}$$

The gradient of the Hankel function in cylindrical coordinates is [4]

$$\nabla H_0^{(2)}(k\rho) = -(\boldsymbol{\rho} - \boldsymbol{\rho}')\frac{k}{\rho}H_1^{(2)}(k\rho) \,, \tag{3.115}$$

which results in

$$\mathbf{H}_A(\boldsymbol{\rho}) = j\frac{k}{4}\int_S (\boldsymbol{\rho} - \boldsymbol{\rho}') \times \mathbf{J}(\boldsymbol{\rho}')\, \frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,. \tag{3.116}$$

Expanding into Cartesian components, the above can be written as

$$\begin{aligned}
H_{A,x}(\boldsymbol{\rho}) &= j\frac{k}{4}\int_S \Big[\Delta_y J_z\Big]\frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,, \\
H_{A,y}(\boldsymbol{\rho}) &= j\frac{k}{4}\int_S \Big[-\Delta_x J_z\Big]\frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,, \\
H_{A,z}(\boldsymbol{\rho}) &= j\frac{k}{4}\int_S \Big[\Delta_x J_y - \Delta_y J_x\Big]\frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,,
\end{aligned} \tag{3.117}$$

where

$$\Delta_x = x - x' \,, \qquad \Delta_y = y - y' \,, \tag{3.118}$$

and

$$\rho = |\boldsymbol{\rho} - \boldsymbol{\rho}'| = \sqrt{(x - x')^2 + (y - y')^2} \,. \tag{3.119}$$

Using (3.2), the corresponding electric field is

$$\begin{aligned}
E_{A,x}(\boldsymbol{\rho}) =& \frac{k^2}{8\omega\epsilon}\int_S \left[\Big([\Delta_y^2-\Delta_x^2]J_x - 2\Delta_x\Delta_y J_y\Big)\frac{H_2^{(2)}(k\rho)}{\rho^2} - J_x H_0^{(2)}(k\rho)\right] d\boldsymbol{\rho}' \,, \\
E_{A,y}(\boldsymbol{\rho}) =& \frac{k^2}{8\omega\epsilon}\int_S \left[\Big([\Delta_x^2-\Delta_y^2]J_y - 2\Delta_x\Delta_y J_x\Big)\frac{H_2^{(2)}(k\rho)}{\rho^2} - J_y H_0^{(2)}(k\rho)\right] d\boldsymbol{\rho}' \,, \\
E_{A,z}(\boldsymbol{\rho}) =& -\frac{k^2}{4\omega\epsilon}\int_S J_z H_0^{(2)}(k\rho)\, d\boldsymbol{\rho}' \,.
\end{aligned} \tag{3.120}$$

Similarly, the electric field radiated by a magnetic current is

$$\mathbf{E}_F(\boldsymbol{\rho}) = -j\frac{k}{4}\int_S (\boldsymbol{\rho}-\boldsymbol{\rho}')\times\mathbf{M}(\boldsymbol{\rho}')\,\frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,, \tag{3.121}$$

and its rectangular components are

$$\begin{aligned}
E_{F,x}(\boldsymbol{\rho}) &= -j\frac{k}{4}\int_S \Big[\Delta_y M_z\Big]\frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,, \\
E_{F,y}(\boldsymbol{\rho}) &= -j\frac{k}{4}\int_S \Big[-\Delta_x M_z\Big]\frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,, \\
E_{F,z}(\boldsymbol{\rho}) &= -j\frac{k}{4}\int_S \Big[\Delta_x M_y - \Delta_y M_x\Big]\frac{H_1^{(2)}(k\rho)}{\rho}\, d\boldsymbol{\rho}' \,.
\end{aligned} \tag{3.122}$$

The corresponding magnetic field is

$$\begin{aligned}
H_{F,x}(\boldsymbol{\rho}) &= \frac{k^2}{8\omega\mu}\int_S \left[\Big([\Delta_y^2-\Delta_x^2]M_x - 2\Delta_x\Delta_y M_y\Big)\frac{H_2^{(2)}(k\rho)}{\rho^2} - M_x H_0^{(2)}(k\rho)\right] d\boldsymbol{\rho}' \,, \\
H_{F,y}(\boldsymbol{\rho}) &= \frac{k^2}{8\omega\mu}\int_S \left[\Big([\Delta_x^2-\Delta_y^2]M_y - 2\Delta_x\Delta_y M_x\Big)\frac{H_2^{(2)}(k\rho)}{\rho^2} - M_y H_0^{(2)}(k\rho)\right] d\boldsymbol{\rho}' \,, \\
H_{F,z}(\boldsymbol{\rho}) &= -\frac{k^2}{4\omega\mu}\int_S M_z H_0^{(2)}(k\rho)\, d\boldsymbol{\rho}' \,.
\end{aligned} \tag{3.123}$$

3.5.3 Three-Dimensional Far Field

When the field point is located very far away from the source, the vectors $\mathbf{r}$ and $\mathbf{r}-\mathbf{r}'$ are virtually parallel, as shown in Figure 3.1b. As a rule of thumb, this is typically satisfied when $kr \gg 1$. In this case, the fields are virtually planar, and under this assumption, we make the following approximations for r in the far field region [1]

$$r = \begin{cases} r - \mathbf{r}'\cdot\hat{\mathbf{r}}\,, & \text{for phase variations,} \\ r\,, & \text{for amplitude variations.} \end{cases} \tag{3.124}$$

Consider the total electric field as expressed in (3.73). We first focus on the components of $\mathbf{E}(\mathbf{r})$ due to $\mathbf{A}(\mathbf{r})$, which are

$$\mathbf{E}_A(\mathbf{r}) = -j\omega\Big[\mathbf{A}(\mathbf{r}) + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{A}(\mathbf{r})\Big] \,. \tag{3.125}$$

The first term on the right-hand side of (3.125) will result in fields that vary according to $1/r$, and the second term in fields that vary according to $1/r^2, 1/r^3$, etc., due to the differential operators. In the far field, we know that only those fields that vary according to $1/r$ are of significant amplitude, and that these components are planar with no components along the direction of propagation. The far electric field $\mathbf{E}_A(\mathbf{r})$ is therefore

$$\mathbf{E}_A(\mathbf{r}) = -j\omega\mathbf{A}(\mathbf{r}) \tag{3.126}$$

and the far magnetic field is

$$\mathbf{H}_F(\mathbf{r}) = -j\omega\mathbf{F}(\mathbf{r}) \,, \tag{3.127}$$

where $\mathbf{E}_A(\mathbf{r})$ and $\mathbf{H}_F(\mathbf{r})$ can be written in terms of their $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ components as

$$\mathbf{E}_A(\mathbf{r}) = -j\omega\big[A_\theta(\mathbf{r})\hat{\boldsymbol{\theta}} + A_\phi(\mathbf{r})\hat{\boldsymbol{\phi}}\big] \tag{3.128}$$

and

$$\mathbf{H}_F(\mathbf{r}) = -j\omega\big[F_\theta(\mathbf{r})\hat{\boldsymbol{\theta}} + F_\phi(\mathbf{r})\hat{\boldsymbol{\phi}}\big] \,. \tag{3.129}$$

To obtain the far electric field $\mathbf{E}_F(\mathbf{r})$ due to $\mathbf{F}(\mathbf{r})$ we can use (3.66), but instead we will use the fact that for plane waves, the electric and magnetic fields are related via the relationship

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\eta}\,\hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r}) \,, \tag{3.130}$$

where $\hat{\mathbf{r}}$ is the direction of propagation and $\eta = \sqrt{\mu/\epsilon}$ is the impedance of the dielectric medium. Using this relationship we can then write

$$\mathbf{E}_F(\mathbf{r}) = -j\omega\eta\big[F_\phi(\mathbf{r})\hat{\boldsymbol{\theta}} - F_\theta(\mathbf{r})\hat{\boldsymbol{\phi}}\big] \tag{3.131}$$

and

$$\mathbf{H}_A(\mathbf{r}) = -j\frac{\omega}{\eta}\big[-A_\phi(\mathbf{r})\hat{\boldsymbol{\theta}} + A_\theta(\mathbf{r})\hat{\boldsymbol{\phi}}\big] \,. \tag{3.132}$$

The total far fields are then

$$\mathbf{E}(\mathbf{r}) = -j\omega\Big(\big[A_\theta(\mathbf{r}) + \eta F_\phi(\mathbf{r})\big]\hat{\boldsymbol{\theta}} + \big[A_\phi(\mathbf{r}) - \eta F_\theta(\mathbf{r})\big]\hat{\boldsymbol{\phi}}\Big) \tag{3.133}$$

and

$$\mathbf{H}(\mathbf{r}) = -j\frac{\omega}{\eta}\Big(\big[-A_\phi(\mathbf{r}) + \eta F_\theta(\mathbf{r})\big]\hat{\boldsymbol{\theta}} + \big[A_\theta(\mathbf{r}) + \eta F_\phi(\mathbf{r})\big]\hat{\boldsymbol{\phi}}\Big) \,, \tag{3.134}$$

where in the far field, $\mathbf{A}(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$ can be written as

$$\mathbf{A}(\mathbf{r}) = \mu\frac{e^{-jkr}}{4\pi r}\int_V \mathbf{J}(\mathbf{r}')e^{jk\mathbf{r}'\cdot\hat{\mathbf{r}}}\,d\mathbf{r}' \tag{3.135}$$

and

$$\mathbf{F}(\mathbf{r}) = \epsilon\frac{e^{-jkr}}{4\pi r}\int_V \mathbf{M}(\mathbf{r}')e^{jk\mathbf{r}'\cdot\hat{\mathbf{r}}}\,d\mathbf{r}' \,. \tag{3.136}$$

3.5.3.1 Scattering Matrix

In scattering problems, the currents **J** and **M** induced by the incident fields $\mathbf{E}^i$ and $\mathbf{H}^i$ generate the scattered far fields $\mathbf{E}^s$ and $\mathbf{H}^s$. To completely characterize the scattered fields, it is sufficient to calculate the field due to incident plane waves having orthogonal polarizations at each incident and scattering angle pair. Because the scattered far field is also planar and has only $\hat{\boldsymbol{\theta}}$- and $\hat{\boldsymbol{\phi}}$-polarized components, it is convenient to use these as the basis pairs for scattering calculations in three dimensions. In this case, the incident fields are given by

$$\mathbf{E}^i(\mathbf{r}) = \left[E^i_\theta \hat{\boldsymbol{\theta}}^i + E^i_\phi \hat{\boldsymbol{\phi}}^i\right] e^{jk_0 \hat{\mathbf{r}}^i \cdot \mathbf{r}} \,, \tag{3.137}$$

and

$$\mathbf{H}^i(\mathbf{r}) = -\frac{1}{\eta_0}\hat{\mathbf{r}}^i \times \mathbf{E}^i(\mathbf{r}) = \frac{1}{\eta_0}\left[-E^i_\theta \hat{\boldsymbol{\phi}}^i + E^i_\phi \hat{\boldsymbol{\theta}}^i\right] e^{jk_0 \hat{\mathbf{r}}^i \cdot \mathbf{r}} \,, \tag{3.138}$$

where the electric field typically has unit amplitude, so $E^i_\theta = E^i_\phi = 1$.

Given the $\hat{\boldsymbol{\theta}}^i$- and $\hat{\boldsymbol{\phi}}^i$-polarized incident electric fields E^i_θ and E^i_ϕ, the $\hat{\boldsymbol{\theta}}^s$- and $\hat{\boldsymbol{\phi}}^s$-polarized scattered electric fields E^s_θ and E^s_ϕ can then be written as

$$\begin{bmatrix} E^s_\theta \\ E^s_\phi \end{bmatrix} = \begin{bmatrix} S_{\theta\theta} & S_{\theta\phi} \\ S_{\phi\theta} & S_{\phi\phi} \end{bmatrix} \begin{bmatrix} E^i_\theta \\ E^i_\phi \end{bmatrix} \,, \tag{3.139}$$

where the matrix **S** is called the *scattering matrix*. For each incident wave, there is a scattered field of the same polarization (co-polarization) and one of the opposite polarization (cross-polarization). The scattering matrix is a common output produced by most radar cross section prediction software codes. It is also common in the literature to see $\hat{\boldsymbol{\theta}}$ polarization referred to as *vertical* polarization (or V-pol) and $\hat{\boldsymbol{\phi}}$ polarization as *horizontal* polarization (or H-pol). We will use this nomenclature when analyzing the RCS of three-dimensional objects in later chapters.

3.5.3.2 Three-Dimensional Radar Cross Section (RCS)

The three-dimensional radar cross section σ_{3D} is defined as [1]

$$\sigma_{3D} = 4\pi r^2 \frac{|\mathbf{E}^s|^2}{|\mathbf{E}^i|^2} \,, \tag{3.140}$$

where typically, $|\mathbf{E}^i| = 1$ for convenience.

3.5.4 Two-Dimensional Far Field

To determine the far field expressions for $\mathbf{A}(\boldsymbol{\rho})$ and $\mathbf{F}(\boldsymbol{\rho})$, we use the large-argument approximation of the Hankel function [4]

$$H^{(2)}_n(k\rho) \approx \sqrt{\frac{2j}{\pi k \rho}}\, j^n e^{-jk\rho} \qquad \rho \to \infty \,, \tag{3.141}$$

and the two-dimensional version of (3.124) to write

$$H_0^{(2)}(k\rho) = \sqrt{\frac{2j}{\pi k\rho}}\, e^{-jk\rho} e^{jk\boldsymbol{\rho}'\cdot\hat{\boldsymbol{\rho}}}\,, \tag{3.142}$$

which yields

$$\mathbf{A}(\boldsymbol{\rho}) = -j\mu\sqrt{\frac{j}{8\pi k}}\,\frac{e^{-jk\rho}}{\sqrt{\rho}}\int_S \mathbf{J}(\boldsymbol{\rho}')\, e^{jk\boldsymbol{\rho}'\cdot\hat{\boldsymbol{\rho}}}\, d\boldsymbol{\rho}' \tag{3.143}$$

and

$$\mathbf{F}(\boldsymbol{\rho}) = -j\epsilon\sqrt{\frac{j}{8\pi k}}\,\frac{e^{-jk\rho}}{\sqrt{\rho}}\int_S \mathbf{M}(\boldsymbol{\rho}')\, e^{jk\boldsymbol{\rho}'\cdot\hat{\boldsymbol{\rho}}}\, d\boldsymbol{\rho}'\,. \tag{3.144}$$

In the two-dimensional case, we know that the total fields $\mathbf{E}(\boldsymbol{\rho})$ and $\mathbf{H}(\boldsymbol{\rho})$ comprise $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\phi}}$ components only. Following similar reasoning as in the three-dimensional case, we can then write the far electric and magnetic fields as

$$\mathbf{E}(\boldsymbol{\rho}) = -j\omega\Big(\big[A_\phi(\boldsymbol{\rho}) + \eta F_z(\boldsymbol{\rho})\big]\hat{\boldsymbol{\phi}} + \big[A_z(\boldsymbol{\rho}) - \eta F_\phi(\boldsymbol{\rho})\big]\hat{\mathbf{z}}\Big) \tag{3.145}$$

and

$$\mathbf{H}(\boldsymbol{\rho}) = -j\frac{\omega}{\eta}\Big(\big[-A_z(\boldsymbol{\rho}) + \eta F_\phi(\boldsymbol{\rho})\big]\hat{\boldsymbol{\phi}} + \big[A_\phi(\boldsymbol{\rho}) + \eta F_z(\boldsymbol{\rho})\big]\hat{\mathbf{z}}\Big)\,. \tag{3.146}$$

3.5.4.1 Two-Dimensional Radar Cross Section (RCS)

The two-dimensional radar cross section σ_{2D} is defined as [1]

$$\sigma_{2D} = 2\pi\rho\,\frac{|\mathbf{E}^s|^2}{|\mathbf{E}^i|^2}\,, \tag{3.147}$$

where typically, $|\mathbf{E}^i| = 1$ for convenience.

3.6 Formulations for Scattering

Scattering problems can be treated as radiation problems where the radiating currents are themselves generated by other currents or fields. For example, to determine the radiation pattern of an antenna, we compute the radiated field due to the current distribution on the antenna. These currents, however, are generated by other sources, such as a voltage applied at the antenna's feedpoint or fields radiated by other antennas nearby. Computation of an object's radar cross section (RCS) is another common example of this problem. In this case, the scattering object (or *target*) is in the far field of an external antenna, and induced currents on the scatterer radiate a secondary or *scattered* field.

Strictly speaking, radiation problems require the integration of known currents $\mathbf{J}$ and $\mathbf{M}$ to obtain the fields by way of either (3.25) and (3.26) or (3.73) and (3.74). Though in some cases the expressions for the currents may be known *a priori*, most practical problems of interest have no analytic solution and the currents must be determined numerically. Thus, these problems usually involve three steps:

1. Formulating an integral equation for the unknown currents $\mathbf{J}$ and $\mathbf{M}$ generated by known fields $\mathbf{E}^i$ or $\mathbf{H}^i$
2. Solving the integral equation
3. Integrating the currents $\mathbf{J}$ and $\mathbf{M}$ to obtain the radiated or scattered fields $\mathbf{E}^s$ and $\mathbf{H}^s$.

The last step is achieved using the methods outlined in Section 3.5 and is relatively straightforward once $\mathbf{J}$ or $\mathbf{M}$ are known. However, the first two steps must be considered carefully, and the remainder of this book is dedicated to this task. We will first review the equivalence principle in Section 3.6.1 and use it to reformulate the scattering problem using equivalent surface currents. Next, we will develop surface integral equations of scattering in Section 3.6.2. We will then apply these integral equations to problems of interest in subsequent chapters, and solve them numerically using the Method of Moments.

3.6.1 Surface Equivalent

The expressions for the radiated fields in (3.73) and (3.74) assume that the currents $\mathbf{J}$ and $\mathbf{M}$ radiate in an unbounded, homogeneous dielectric region with parameters (μ, ϵ). If we now introduce an obstacle into this region with different material properties, we can no longer use (3.73) and (3.74) to determine the radiated fields. This is a serious problem, as virtually all problems of practical interest involve currents radiating in the presence of an obstacle. Fortunately, it is possible to reformulate these problems in terms of an *equivalent* problem that is more convenient to solve in the region of interest. By introducing certain fictitious electric and magnetic currents, we can create a set of coupled integral equations that eliminate the obstacle. Once the equations are solved for these currents, it is straightforward to determine the field in each region. As we treat conducting, homogeneous dielectric and composite conducting/dielectric objects

in this book, we will formulate our equivalent problem in terms of surface currents that exist on the interfaces between dielectric and conducting regions.

The surface equivalence theorem (also called *Huygens' principle*) states that every point on an advancing wavefront is itself a source of radiated waves. Using this theorem, a radiating source can be replaced by a fictitious set of different but *equivalent* sources that lie on an arbitrary closed surface around the original source. By matching the appropriate boundary conditions, these currents will generate the same radiated field outside the closed surface as the original sources. The theorem allows us to determine the far fields radiated by a structure if the near fields are known, or to formulate a surface integral equation that can be solved for the currents induced on an object by an externally incident field.

To introduce the concept, we will develop the integral equations for a small, conceptual scattering problem, and generalize them to general problems later. To begin, consider the geometry of Figure 3.2a, where the electric and magnetic currents $\mathbf{J}$ and $\mathbf{M}$ generate known *incident* fields $\mathbf{E}_1^i$ and $\mathbf{H}_1^i$ in an unbounded region R_1 with dielectric parameters (μ_1, ϵ_1). A small, bounded region R_2 with the same parameters is placed inside R_1, where the interface between the two regions comprises the surface S with normal $\hat{\mathbf{n}}_1$ that points into R_1. Because R_1 and R_2 have the same dielectric parameters, $\mathbf{E}_1^i$ and $\mathbf{H}_1^i$ comprise the fields everywhere. If we now change the parameters of R_2 to (μ_2, ϵ_2) as in Figure 3.2b, it becomes an obstacle inside R_1 and the fields are no longer the same. The fields in R_1 now comprise $\mathbf{E}_1$ and $\mathbf{H}_1$, given by

$$\mathbf{E}_1 = \mathbf{E}_1^i + \mathbf{E}_1^s \tag{3.148}$$

and

$$\mathbf{H}_1 = \mathbf{H}_1^i + \mathbf{H}_1^s \,, \tag{3.149}$$

where $\mathbf{E}_1^s$ and $\mathbf{H}_1^s$ are the perturbed or *scattered* fields generated by the obstacle, and the fields $\mathbf{E}_2$ and $\mathbf{H}_2$ inside R_2 are referred to as *transmitted* fields. Let us now remove $\mathbf{J}$ and $\mathbf{M}$ and replace them by fictitious but *equivalent* surface currents $\mathbf{J}_1$ and $\mathbf{M}_1$ on the outside of S, which we refer to as S^+. For the fields in R_1 and R_2 to remain unchanged, the boundary conditions (Section 3.2) on S^+ must satisfy

$$\mathbf{M}_1 = -\hat{\mathbf{n}}_1 \times (\mathbf{E}_1 - \mathbf{E}_2) \tag{3.150}$$

and

$$\mathbf{J}_1 = \hat{\mathbf{n}}_1 \times (\mathbf{H}_1 - \mathbf{H}_2) \,. \tag{3.151}$$

Note that because the obstacle is still present, we cannot use (3.73) and (3.74) in either (3.150) or (3.151). However, the quantities of interest in scattering problems are usually the fields $\mathbf{E}_1^s$ and $\mathbf{H}_1^s$, not the fields inside the obstacle. Therefore, we apply the *extinction theorem* [2] and simply assume that those fields are zero. Doing so, we can now assign to R_2 the same dielectric parameters as R_1, and the boundary conditions on S^+ become

$$\mathbf{M}_1 = -\hat{\mathbf{n}}_1 \times \mathbf{E}_1 = -\hat{\mathbf{n}}_1 \times (\mathbf{E}_1^s + \mathbf{E}_1^i) \tag{3.152}$$

and

$$\mathbf{J}_1 = \hat{\mathbf{n}}_1 \times \mathbf{H}_1 = \hat{\mathbf{n}}_1 \times (\mathbf{H}_1^s + \mathbf{H}_1^i) \,. \tag{3.153}$$

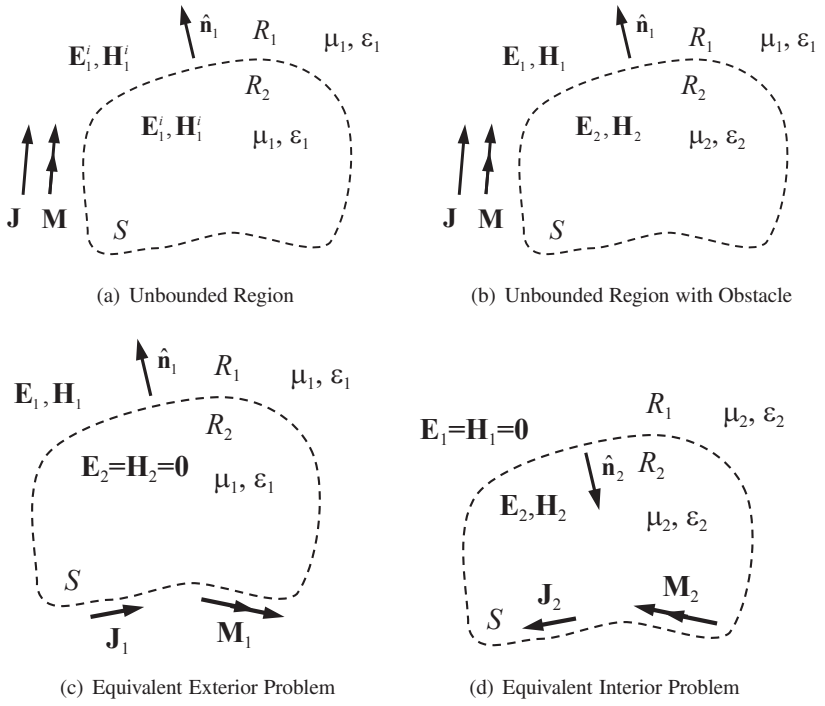


Figure 3.2: Surface Equivalent

We now have an equivalent *exterior* problem, as illustrated in Figure 3.2c, where the fields outside S remain unchanged. We can likewise formulate an equivalent *interior* problem inside R_2. To do so, we place the surface currents $\mathbf{J}_2$ and $\mathbf{M}_2$ on the inside of S, which we refer to as S^-. As before, if we assume that the fields in R_1 are zero, the boundary conditions on S^- become

$$\mathbf{M}_2 = -\hat{\mathbf{n}}_2 \times \mathbf{E}_2 \tag{3.154}$$

and

$$\mathbf{J}_2 = \hat{\mathbf{n}}_2 \times \mathbf{H}_2 \tag{3.155}$$

where $\hat{\mathbf{n}}_2$ points into R_2.

Let us focus again on the fields in R_1. Substituting (3.75) and (3.76) into (3.152) and (3.153) yields

$$\mathbf{M}_1(\mathbf{r}) + \hat{\mathbf{n}}_1(\mathbf{r}) \times \big[-j\omega\mu_1(\mathcal{L}\mathbf{J}_1)(\mathbf{r}) - (\mathcal{K}\mathbf{M}_1)(\mathbf{r}) \big] = -\hat{\mathbf{n}}_1(\mathbf{r}) \times \mathbf{E}_1^i(\mathbf{r}) \tag{3.156}$$

and

$$\mathbf{J}_1(\mathbf{r}) - \hat{\mathbf{n}}_1(\mathbf{r}) \times \big[-j\omega\epsilon_1(\mathcal{L}\mathbf{M}_1)(\mathbf{r}) + (\mathcal{K}\mathbf{J}_1)(\mathbf{r}) \big] = \hat{\mathbf{n}}_1(\mathbf{r}) \times \mathbf{H}_1^i(\mathbf{r}) \,. \tag{3.157}$$

As $\mathbf{r}$ approaches $\mathbf{r}'$ on S^+, the $\mathcal{L}$ operator is well behaved, however the $\mathcal{K}$ operator contains a singularity which must be addressed [2]. To do so, we must determine the value of

$$\lim_{\mathbf{r}\to\mathbf{r}'}\Big[\hat{\mathbf{n}}(\mathbf{r})\times(\mathcal{K}\mathbf{X})(\mathbf{r})\Big]\,. \tag{3.158}$$

Using (3.78), this is

$$\lim_{\mathbf{r}\to\mathbf{r}'}\Big[\hat{\mathbf{n}}(\mathbf{r})\times\nabla\times\int_S G(\mathbf{r},\mathbf{r}')\,\mathbf{X}(\mathbf{r}')\,d\mathbf{r}'\Big]\,, \tag{3.159}$$

and following (3.96), we can move the gradient inside, which yields

$$\lim_{\mathbf{r}\to\mathbf{r}'}\Big[-\hat{\mathbf{n}}(\mathbf{r})\times\int_S \mathbf{X}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'\Big]\,. \tag{3.160}$$

We now break the integral into two parts following [7]

$$-\hat{\mathbf{n}}(\mathbf{r})\times\left[\int_{S-\delta S}\mathbf{X}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'+\int_{\delta S}\mathbf{X}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'\right], \tag{3.161}$$

where δS is a very small, circular region of radius a in S about $\mathbf{r}$. If we now make the center of δS the origin of a local cylindrical coordinate system, we can write

$$|\mathbf{r}-\mathbf{r}'|=\sqrt{(\rho')^2+z^2}\,, \tag{3.162}$$

and for $|\mathbf{r}-\mathbf{r}'|\ll 1$, the Green's function inside δS can be written as

$$G(\mathbf{r},\mathbf{r}')=\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\approx\frac{1}{4\pi\sqrt{(\rho')^2+(z-z')^2}}\,. \tag{3.163}$$

The gradient in cylindrical coordinates is

$$\nabla=\frac{\partial}{\partial\rho}\hat{\boldsymbol{\rho}}+\frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\,, \tag{3.164}$$

and because $\hat{\mathbf{n}}(\mathbf{r})=\hat{\mathbf{z}}$ on δS and $\mathbf{X}(\mathbf{r}')$ is everywhere tangent to δS, we can write

$$\hat{\mathbf{z}}\times\mathbf{X}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')=\mathbf{X}(\mathbf{r}')\Big[\frac{\partial}{\partial z}G(\mathbf{r},\mathbf{r}')\Big]\,, \tag{3.165}$$

which is

$$\mathbf{X}(\mathbf{r}')\Big[\frac{\partial}{\partial z}G(\mathbf{r},\mathbf{r}')\Big]=\mathbf{X}(\mathbf{r}')\frac{-z}{4\pi\big[(\rho')^2+z^2\big]^{3/2}}\,. \tag{3.166}$$

Because δS is very small, we will assume that $\mathbf{X}(\mathbf{r}')$ is approximately equal to $\mathbf{X}(\mathbf{r})$. We now write the integral over δS as

$$-\hat{\mathbf{n}}(\mathbf{r})\times\int_{\delta S}\mathbf{X}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'=\frac{\mathbf{X}(\mathbf{r})}{2}\int_0^a\frac{z\rho'}{\big[(\rho')^2+z^2\big]^{3/2}}\,d\rho'\,. \tag{3.167}$$

Integrating this expression we get

$$\frac{\mathbf{X}(\mathbf{r})}{2}\left[\frac{z}{|z|}-\frac{z}{\sqrt{a^2+z^2}}\right], \tag{3.168}$$

and taking the limit as z approaches 0 from above yields

$$\lim_{z\to 0^+}\frac{\mathbf{X}(\mathbf{r})}{2}\left[\frac{z}{|z|}-\frac{z}{\sqrt{a^2+z^2}}\right]=\frac{\mathbf{X}(\mathbf{r})}{2}. \tag{3.169}$$

Thus,

$$\lim_{\mathbf{r}\to\mathbf{r}'}\Big[\hat{\mathbf{n}}_1(\mathbf{r})\times(\mathcal{K}\mathbf{X})(\mathbf{r})\Big]=\frac{\mathbf{X}(\mathbf{r})}{2}. \tag{3.170}$$

Note that in cases where the small area δS is not locally planar, such as the apex of a cone or an edge shared by two planar facets, a modification of (3.170) is required. It is shown in [7] that the result of (3.170) should be adjusted to take into account the exterior solid angle Ω_0, which results in

$$\lim_{\mathbf{r}\to\mathbf{r}'}\Big[\hat{\mathbf{n}}_1(\mathbf{r})\times(\mathcal{K}\mathbf{X})(\mathbf{r})\Big]=\frac{\Omega_0}{4\pi}\mathbf{X}(\mathbf{r}). \tag{3.171}$$

For smooth geometries, the exterior solid angle is $\Omega_0=2\pi$ at every observation point, and (3.171) reduces to (3.170). When an object has sharp corners or edges, the solid angle can be adjusted as in [8] for three-dimensional objects (see Chapter 8). However, in this book we will use (3.170) with no modifications, which is a common approach in the literature. Thus, (3.156) and (3.157) become

$$\frac{1}{2}\mathbf{M}_1(\mathbf{r})+\hat{\mathbf{n}}_1(\mathbf{r})\times\big[-j\omega\mu_1(\mathcal{L}\mathbf{J}_1)(\mathbf{r})-(\mathcal{K}\mathbf{M}_1)(\mathbf{r})\big]=-\hat{\mathbf{n}}_1(\mathbf{r})\times\mathbf{E}_1^i(\mathbf{r}) \tag{3.172}$$

and

$$\frac{1}{2}\mathbf{J}_1(\mathbf{r})-\hat{\mathbf{n}}_1(\mathbf{r})\times\big[-j\omega\epsilon_1(\mathcal{L}\mathbf{M}_1)(\mathbf{r})+(\mathcal{K}\mathbf{J}_1)(\mathbf{r})\big]=\hat{\mathbf{n}}_1(\mathbf{r})\times\mathbf{H}_1^i(\mathbf{r}) \tag{3.173}$$

for $\mathbf{r}$ on S^+, where the $\frac{1}{2}\mathbf{M}_1(\mathbf{r})$ and $\frac{1}{2}\mathbf{J}_1(\mathbf{r})$ terms are used when $\mathbf{r}\to\mathbf{r}'$, and the $(\mathcal{K}\mathbf{X})(\mathbf{r})$ operators are used otherwise. Similarly, the fields in R_2 can be written as

$$\frac{1}{2}\mathbf{M}_2(\mathbf{r})+\hat{\mathbf{n}}_2(\mathbf{r})\times\big[-j\omega\mu_2(\mathcal{L}\mathbf{J}_2)(\mathbf{r})-(\mathcal{K}\mathbf{M}_2)(\mathbf{r})\big]=0 \tag{3.174}$$

and

$$\frac{1}{2}\mathbf{J}_2(\mathbf{r})-\hat{\mathbf{n}}_2(\mathbf{r})\times\big[-j\omega\epsilon_2(\mathcal{L}\mathbf{M}_2)(\mathbf{r})+(\mathcal{K}\mathbf{J}_2)(\mathbf{r})\big]=0 \tag{3.175}$$

for $\mathbf{r}$ on S^-. Note that since a limiting argument was used to determine the value of $(\mathcal{K}\mathbf{X})(\mathbf{r})$ as $\mathbf{r}\to\mathbf{r}'$ on $S^{\pm}$, it is only valid for surfaces that are closed and cannot be used on open surfaces or those that are very thin.

We can now state the surface integral equations for the conceptual problem outlined in Section 3.6.1. Using the fact that on S,

$$\hat{\mathbf{n}}\times\hat{\mathbf{n}}\times\mathbf{A}=(\hat{\mathbf{n}}\cdot\mathbf{A})\hat{\mathbf{n}}-\mathbf{A}=-\mathbf{A}_{tan}, \tag{3.176}$$

we can write (3.172) and (3.173) as

$$\left[j\omega\mu_1(\mathcal{L}\mathbf{J}_1)(\mathbf{r}) + (\mathcal{K}\mathbf{M}_1)(\mathbf{r})\right]_{tan} + \frac{1}{2}\hat{\mathbf{n}}_1(\mathbf{r}) \times \mathbf{M}_1(\mathbf{r}) = \left[\mathbf{E}_1^i(\mathbf{r})\right]_{tan} \tag{3.177}$$

and

$$\left[j\omega\epsilon_1(\mathcal{L}\mathbf{M}_1)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_1)(\mathbf{r})\right]_{tan} - \frac{1}{2}\hat{\mathbf{n}}_1(\mathbf{r}) \times \mathbf{J}_1(\mathbf{r}) = \left[\mathbf{H}_1^i(\mathbf{r})\right]_{tan}, \tag{3.178}$$

and (3.174) and (3.175) as

$$\left[j\omega\mu_2(\mathcal{L}\mathbf{J}_2)(\mathbf{r}) + (\mathcal{K}\mathbf{M}_2)(\mathbf{r})\right]_{tan} + \frac{1}{2}\hat{\mathbf{n}}_2(\mathbf{r}) \times \mathbf{M}_2(\mathbf{r}) = 0 \tag{3.179}$$

and

$$\left[j\omega\epsilon_2(\mathcal{L}\mathbf{M}_2)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_2)(\mathbf{r})\right]_{tan} - \frac{1}{2}\hat{\mathbf{n}}_2(\mathbf{r}) \times \mathbf{J}_2(\mathbf{r}) = 0\,. \tag{3.180}$$

Equations (3.177–3.180) comprise a set of integral equations for the unknown currents $\mathbf{J}_1, \mathbf{M}_1, \mathbf{J}_2$ and $\mathbf{M}_2$, in terms of the known incident fields $\mathbf{E}_1^i$ and $\mathbf{H}_1^i$ in R_1. Once the currents are found, the fields everywhere in R_1 and R_2 can be found using (3.75) and (3.76). As written, these equations do not appear coupled, as they have no quantities in common. However, the currents on each side of S are related via the boundary conditions. We will consider this in detail in the next section.

3.6.2 Surface Integral Equations

Let us now replace the simple scatterer from Section 3.6.1 with an object of a more general configuration, as shown in Figure 3.3. This object comprises multiple, piecewise homogeneous dielectric (penetrable) or conducting (PEC) regions $R_1, \ldots, R_N$. The material parameters of dielectric region R_l are (μ_l, ϵ_l), where either μ or ϵ may be complex (denoting a lossy material) and R_0 is the unbounded external region to which we assign the free space parameters (μ_0, ϵ_0). The boundary of R_l comprises the surface S_l, and the interface between regions R_l and R_m is the surface S_{lm} (or S_{ml}), which may comprise a portion or all of S_l. The normal vector $\hat{\mathbf{n}}_l$ points into R_l on each of its interfaces. Conductors may be closed regions with finite volume (R_4 in Figure 3.3), or open, infinitely thin sheets. For each non-PEC region R_l, the surface integral equations on S_l are

$$\left[j\omega\mu_l(\mathcal{L}\mathbf{J}_l)(\mathbf{r}) + (\mathcal{K}\mathbf{M}_l)(\mathbf{r})\right]_{tan} + \frac{1}{2}\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{M}_l(\mathbf{r}) = \left[\mathbf{E}_l^i(\mathbf{r})\right]_{tan} \tag{3.181}$$

and

$$\left[j\omega\epsilon_l(\mathcal{L}\mathbf{M}_l)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_l)(\mathbf{r})\right]_{tan} - \frac{1}{2}\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{J}_l(\mathbf{r}) = \left[\mathbf{H}_l^i(\mathbf{r})\right]_{tan}. \tag{3.182}$$

For scattering problems, $\mathbf{E}_l^i(\mathbf{r})$ and $\mathbf{H}_l^i(\mathbf{r})$ are zero everywhere except in R_0. For antenna and radiation problems, $\mathbf{E}_l^i(\mathbf{r})$ and $\mathbf{H}_l^i(\mathbf{r})$ may be nonzero in one or more regions, depending on where the excitation is applied. We refer to (3.181) as the Electric Field Integral Equation (EFIE), and to (3.182) as the Magnetic Field Integral Equation (MFIE).

3.6.2.1 Interior Resonance Problem

Applying (3.181) and (3.182) to dielectrics will yield a result free of spurious solutions [9], however on closed conductors the EFIE and MFIE cannot produce a unique solution for all frequencies [9, 10, 11, 12, 13]. This is because there are homogeneous solutions to these equations that satisfy the boundary conditions with zero incident field. These spurious solutions correspond to interior cavity (resonant) modes of the object itself, and radiate no field outside the object. One method for dealing with this problem is through the application of an extended boundary condition, where the original integral equations are modified or augmented such that the value of the field inside the object is explicitly forced to be zero [7, 14]. Dual-surface electric and magnetic integral equations (DSEFIE, DSMFIE) [15] are an example of such an extended boundary condition. In such a formulation, a second surface is placed just inside the original surface. The appropriate boundary condition for the internal fields on this surface is used to generate an additional integral equation. The combination of this new equation with the original results in a combined equation that produces a unique solution at all frequencies. This method, however, requires additional effort in generating the secondary surface, and the number of currents is doubled, which drives up the demand for CPU time and system memory.

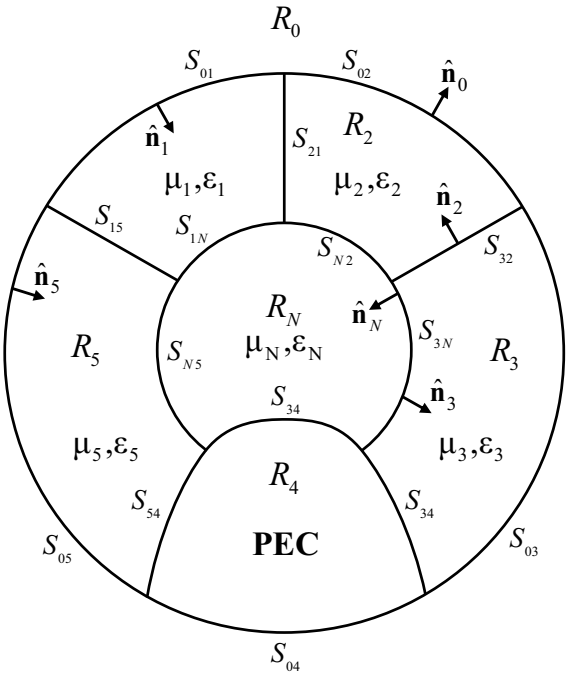


Figure 3.3: Composite Multi-Region Geometry

3.6.2.1.1 Combined Field Integral Equation

To date, the most popular method for handling the resonance problem is a linear combination of the EFIE and a modified MFIE. This enforces the boundary conditions on the electric and magnetic field simultaneously and is free of spurious solutions, as the null spaces of the EFIE and the modified MFIE differ. To create the modified MFIE, we take the cross product of $\hat{\mathbf{n}}_l$ and (3.182), yielding

$$\hat{\mathbf{n}}_l \times \left[j\omega\epsilon_l(\mathcal{L}\mathbf{M}_l)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_l)(\mathbf{r}) \right]_{tan} + \frac{1}{2}\mathbf{J}_l(\mathbf{r}) = \hat{\mathbf{n}}_l \times \left[\mathbf{H}_l^i(\mathbf{r}) \right]_{tan} , \tag{3.183}$$

where there are now $\hat{\mathbf{n}} \times \mathcal{L}$ and $\hat{\mathbf{n}} \times \mathcal{K}$ operators. Herein, we will refer to (3.183) as the nMFIE, which when combined with the EFIE yields the Combined Field Integral Equation (CFIE) given by [9]

$$\alpha\text{EFIE} + (1-\alpha)\eta_l\text{nMFIE} , \tag{3.184}$$

where $\eta_l = \sqrt{\mu_l/\epsilon_l}$ and $0 \leq \alpha \leq 1$, with $\alpha = 0.5$ commonly used. We apply the CFIE to closed, conducting volumes in this book. For open, thin conductors, we will use the EFIE only ($\alpha = 1$).

3.6.2.1.2 A Note on Nomenclature

Many other books and journal papers that discuss surface integral equations are concerned with conducting (PEC) objects only. The equations in these references are typically the EFIE (3.181) and the nMFIE of (3.183) with no magnetic currents, except that the nMFIE is often referred to as simply the MFIE. References that treat conducting as well as dielectric objects will more often use the nomenclature outlined in this book.

3.6.2.2 Discretization and Testing

We will now use the Method of Moments to convert (3.181), (3.182), and (3.183) into a matrix system using Galerkin-type testing in each region. Expanding the electric and magnetic currents in R_l using a sum of weighted basis functions yields

$$\mathbf{J}_l(\mathbf{r}) = \sum_{n=1}^{N_J^{(l)}} I_n^{J(l)} \mathbf{f}_n(\mathbf{r}) \tag{3.185}$$

and

$$\mathbf{M}_l(\mathbf{r}) = \sum_{n=1}^{N_M^{(l)}} I_n^{M(l)} \mathbf{g}_n(\mathbf{r}) , \tag{3.186}$$

where $N_J^{(l)}$ and $N_M^{(l)}$ are the number of electric and magnetic basis functions in R_l, respectively, and the basis functions $\mathbf{f}_n(\mathbf{r})$ and $\mathbf{g}_n(\mathbf{r})$ will remain undefined for now.

We next test the EFIE and nMFIE with the electric testing functions $\mathbf{f}_m(\mathbf{r})$, and the MFIE with the magnetic testing functions $\mathbf{g}_m(\mathbf{r})$ as in [9]. This yields in R_l the matrix system

$$\begin{bmatrix} Z^{EJ(l)} & Z^{EM(l)} \\ Z^{HJ(l)} & Z^{HM(l)} \\ Z^{nHJ(l)} & Z^{nHM(l)} \end{bmatrix} \begin{bmatrix} I^{J(l)} \\ I^{M(l)} \end{bmatrix} = \begin{bmatrix} V^{E(l)} \\ V^{H(l)} \\ V^{nH(l)} \end{bmatrix}, \tag{3.187}$$

with matrix elements given by

$$\begin{aligned} Z_{mn}^{EJ(l)} &= j\omega\mu_l \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \int_{\mathbf{f}_n} \mathbf{f}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r} \\ &\quad + \frac{j}{\omega\epsilon_l} \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \Big[\nabla\nabla \cdot \int_{\mathbf{f}_n} \mathbf{f}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\Big]\, d\mathbf{r}\,, \end{aligned} \tag{3.188}$$

$$\begin{aligned} Z_{mn}^{EM(l)} &= \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \int_{\mathbf{g}_n} \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{g}_n(\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r} \\ &\quad + \frac{1}{2} \int_{\mathbf{f}_m,\mathbf{g}_n} \mathbf{f}_m(\mathbf{r}) \cdot \Big[\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{g}_n(\mathbf{r})\Big]\, d\mathbf{r}\,, \end{aligned} \tag{3.189}$$

$$\begin{aligned} Z_{mn}^{HJ(l)} &= -\int_{\mathbf{g}_m} \mathbf{g}_m(\mathbf{r}) \cdot \int_{\mathbf{f}_n} \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r} \\ &\quad - \frac{1}{2} \int_{\mathbf{g}_m,\mathbf{f}_n} \mathbf{g}_m(\mathbf{r}) \cdot \Big[\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{f}_n(\mathbf{r})\Big]\, d\mathbf{r}\,, \end{aligned} \tag{3.190}$$

$$\begin{aligned} Z_{mn}^{HM(l)} &= j\omega\epsilon_l \int_{\mathbf{g}_m} \mathbf{g}_m(\mathbf{r}) \cdot \int_{\mathbf{g}_n} \mathbf{g}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r} \\ &\quad + \frac{j}{\omega\mu_l} \int_{\mathbf{g}_m} \mathbf{g}_m(\mathbf{r}) \cdot \Big[\nabla\nabla \cdot \int_{\mathbf{g}_n} \mathbf{g}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\Big]\, d\mathbf{r}\,, \end{aligned} \tag{3.191}$$

$$\begin{aligned} Z_{mn}^{nHJ(l)} &= -\int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \Big[\hat{\mathbf{n}}_l(\mathbf{r}) \times \int_{\mathbf{f}_n} \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}')\, d\mathbf{r}'\Big]\, d\mathbf{r} \\ &\quad + \frac{1}{2} \int_{\mathbf{f}_m,\mathbf{f}_n} \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{f}_n(\mathbf{r})\, d\mathbf{r}\,, \end{aligned} \tag{3.192}$$

$$\begin{aligned} Z_{mn}^{nHM(l)} &= j\omega\epsilon_l \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \Big[\hat{\mathbf{n}}_l(\mathbf{r}) \times \int_{\mathbf{g}_n} \mathbf{g}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\Big]\, d\mathbf{r} \\ &\quad + \frac{j}{\omega\mu_l} \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \Big[\hat{\mathbf{n}}_l(\mathbf{r}) \times \nabla\nabla \cdot \int_{\mathbf{g}_n} \mathbf{g}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\Big]\, d\mathbf{r}\,, \end{aligned} \tag{3.193}$$

where the $\mathcal{K}$ and $\hat{\mathbf{n}} \times \mathcal{K}$ operators were modified following (3.96). The elements of the right-hand side vectors are

$$V_m^{E(l)} = \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{E}_l^i(\mathbf{r})\, d\mathbf{r}\,, \tag{3.194}$$

$$V_m^{H(l)} = \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{H}_l^i(\mathbf{r})\, d\mathbf{r}\,, \tag{3.195}$$

and

$$V_m^{nH(l)} = \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \left[\hat{\mathbf{n}}_l \times \mathbf{H}_l^i(\mathbf{r})\right] d\mathbf{r}\,. \tag{3.196}$$

Note that all blocks of the matrix in (3.187) are square. If we now combine the matrix equations for all regions, we obtain a block-diagonal system of the form

$$\begin{bmatrix} Z^{(1)} & 0 & \cdots & 0 \\ 0 & Z^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & Z^{(N)} \end{bmatrix} \begin{bmatrix} I^{(1)} \\ I^{(2)} \\ \vdots \\ I^{(N)} \end{bmatrix} = \begin{bmatrix} V^{(1)} \\ V^{(2)} \\ \vdots \\ V^{(N)} \end{bmatrix}, \tag{3.197}$$

where

$$Z^{(l)} = \begin{bmatrix} Z^{EJ(l)} & Z^{EM(l)} \\ Z^{HJ(l)} & Z^{HM(l)} \\ Z^{nHJ(l)} & Z^{nHM(l)} \end{bmatrix}, \tag{3.198}$$

$$I^{(l)} = \begin{bmatrix} I^{J(l)} \\ I^{M(l)} \end{bmatrix}, \tag{3.199}$$

and

$$V^{(l)} = \begin{bmatrix} V^{E(l)} \\ V^{H(l)} \\ V^{nH(l)} \end{bmatrix}. \tag{3.200}$$

Note that when treating conductors, the second block row in (3.187), corresponding to the MFIE, will not exist. Likewise, the third block row exists only when the CFIE is used on closed conductors.

3.6.2.3 Modification of Matrix Elements

If the basis and testing functions have a well-behaved divergence, it is possible to redistribute the vector derivatives in the $\mathcal{L}$ operator so that they do not operate on the Green's function. This is advantageous as they would otherwise compound the $1/r$ singularity. To do so, let us concentrate on the expression having the form

$$\int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \left[\nabla\nabla \cdot \int_{\mathbf{f}_n} \mathbf{f}_n(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\right] d\mathbf{r}\,. \tag{3.201}$$

Following Section 3.4.4, this can be rewritten as

$$\int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \nabla S(\mathbf{r})\, d\mathbf{r} = \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \left[\nabla \int_{\mathbf{f}_n} \nabla' \cdot \mathbf{f}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\right] d\mathbf{r}\,. \quad (3.202)$$

Using the vector identity

$$\mathbf{f}(\mathbf{r}) \cdot \nabla S(\mathbf{r}) = \nabla \cdot \big[\mathbf{f}(\mathbf{r}) S(\mathbf{r})\big] - \big[\nabla \cdot \mathbf{f}(\mathbf{r})\big] S(\mathbf{r})\,, \quad (3.203)$$

we can write the above as

$$\int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \nabla S(\mathbf{r})\, d\mathbf{r} = \int_{\mathbf{f}_m} \nabla \cdot \big[\mathbf{f}_m(\mathbf{r}) S(\mathbf{r})\big]\, d\mathbf{r} - \int_{\mathbf{f}_m} \big[\nabla \cdot \mathbf{f}_m(\mathbf{r})\big] S(\mathbf{r})\, d\mathbf{r}\,. \quad (3.204)$$

We convert the first term on the right-hand side to a surface integral using the divergence theorem

$$\int_V \nabla \cdot \big[\mathbf{f}_m(\mathbf{r}) S(\mathbf{r})\big]\, d\mathbf{r} = \int_S \hat{\mathbf{n}} \cdot \big[\mathbf{f}_m(\mathbf{r}) S(\mathbf{r})\big]\, d\mathbf{r}\,. \quad (3.205)$$

Since the bounding surface can be made large enough that $\mathbf{f}_m(\mathbf{r})$ vanishes, the above goes to zero leaving only the second term, which is

$$\int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \nabla S(\mathbf{r})\, d\mathbf{r} = -\int_{\mathbf{f}_m} \nabla \cdot \mathbf{f}_m(\mathbf{r}) \int_{\mathbf{f}_n} \nabla' \cdot \mathbf{f}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r}\,. \quad (3.206)$$

Substitution of the above into (3.188) yields

$$\begin{aligned} Z_{mn}^{EJ(l)} = {} & j\omega\mu_l \int_{\mathbf{f}_m} \mathbf{f}_m(\mathbf{r}) \cdot \int_{\mathbf{f}_n} \mathbf{f}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r} \\ & - \frac{j}{\omega\epsilon_l} \int_{\mathbf{f}_m} \nabla \cdot \mathbf{f}_m(\mathbf{r}) \int_{\mathbf{f}_n} \nabla' \cdot \mathbf{f}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r}\,, \end{aligned} \quad (3.207)$$

and a similar substitution can also be made in (3.191).

Using the vector identity $\mathbf{a} \cdot \big(\mathbf{b} \times \mathbf{c}\big) = \big(\mathbf{a} \times \mathbf{b}\big) \cdot \mathbf{c}$, and the result from (3.207), we can also write the second integral in (3.193) as

$$-\int_{\mathbf{f}_m} \nabla \cdot \Big[\mathbf{f}_m(\mathbf{r}) \times \hat{\mathbf{n}}_l(\mathbf{r})\Big] \int_{\mathbf{g}_n} \nabla' \cdot \mathbf{g}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r}\,. \quad (3.208)$$

Using the divergence theorem, the outermost integral can be converted to a contour integral, which yields

$$-\oint_{\delta\mathbf{f}_m} \hat{\mathbf{c}}_m(\mathbf{r}) \cdot \Big[\mathbf{f}_m(\mathbf{r}) \times \hat{\mathbf{n}}_l(\mathbf{r})\Big] \int_{\mathbf{g}_n} \nabla' \cdot \mathbf{g}_n(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r}\,. \quad (3.209)$$

This integral is now performed along the boundary of the support of $\mathbf{f}_m(\mathbf{r})$, where $\hat{\mathbf{c}}_m(\mathbf{r})$ is the outward-facing normal to that boundary. The choice of (3.208) or (3.209) in the implementation of (3.193) depends on the geometry and testing functions, and we will use both forms in this book.

3.6.3 Enforcement of Boundary Conditions

The block-diagonal system in (3.197) was defined region-wise without regard to the boundary conditions on the interfaces between regions. We know from (3.5) and (3.6) that the tangential components of the electric and magnetic fields are discontinuous across a dielectric interface by an amount equal to the magnetic and electric surface currents on that interface, respectively. However, in our integral equation formulation there are no real surface currents on dielectric interfaces; they are fictitious. This implies that at the interface between dielectric regions R_l and R_m,

$$\mathbf{J}_l(\mathbf{r}) = -\mathbf{J}_m(\mathbf{r}) \,, \tag{3.210}$$

$$\mathbf{M}_l(\mathbf{r}) = -\mathbf{M}_m(\mathbf{r}) \,, \tag{3.211}$$

and therefore, (3.197) contains linearly dependent unknowns. This can be addressed by assigning an opposing orientation to the basis functions on opposite sides of an interface, so that they will have the same coefficient. As these basis functions now represent a single unknown, the corresponding columns of (3.197) can be combined. Note that by doing this, (3.197) becomes overdetermined as it has more rows than columns, and so we must address that problem as well.

3.6.3.1 EFIE-CFIE-PMCHWT Approach

The overdetermined system can be handled in several ways, though the most common approach is the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) formulation [16, 17]. In this method, the EFIEs and MFIEs on each side of dielectric interfaces are added together by combining the corresponding rows of the matrix. Doing so, (3.197) becomes square, and can be solved. We can combine this with the treatment for open and closed conductors (EFIE and CFIE, respectively) to yield a unified approach for composite objects that we refer to as the EFIE-CFIE-PMCHWT formulation [9]. This approach can be summarized as follows:

1. On dielectric interfaces we test the EFIE and MFIE, and eliminate linearly dependent unknowns using PMCHWT.
2. On interfaces between a dielectric and a closed conductor, we test the CFIE.
3. On interfaces between a dielectric and an open conductor, we test the EFIE only.

3.6.3.1.1 Treatment at Junctions Points where three or more interfaces meet are referred to as *junctions*. Enforcement of the EFIE-CFIE-PMCHWT formulation at junctions requires special consideration for specific problem types. We will consider junctions in more detail in later chapters.

3.7 Other Interface Types

The EFIE and MFIE as presented in Section 3.6.2 were derived for interfaces between dielectric regions, and the boundary conditions enforced on those interfaces following

the procedure in Section 3.6.3. There are two other interface types which appear often in surface integral equations, the Leontovich Impedance Boundary Condition (IBC) and the resistive boundary (often referred to as a "resistive card" or "r-card"). In this section, we will derive the surface integral equations for each of these interface types.

3.7.1 Leontovich Impedance Boundary Condition (IBC)

The Leontovich IBC is a simple approximation, often used to simplify the calculation of scattering from electrically large scatterers, which are most often penetrable and possibly lossy. This boundary condition is applied to a scatterer's exterior surface, where it replaces the scattering physics on the interior with an effective surface impedance. This boundary condition relates the electric and magnetic field on the exterior surface as [18, 19]

$$\mathbf{E}_{tan} = -\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E} = Z\hat{\mathbf{n}} \times \mathbf{H}\,, \tag{3.212}$$

where Z is the effective surface impedance, measured in Ohms per square (Ω/sq) and

$$\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{J} \tag{3.213}$$

and

$$\hat{\mathbf{n}} \times \mathbf{E} = -\mathbf{M}. \tag{3.214}$$

The value of Z is dependent on the structure on the interior of the boundary, and while Z is frequency dependent, it has no angle component. Thus, values of Z are typically chosen at broadside incidence. For some types of structures, and at high frequencies where the local surface curvature is much less than the wavelength, this approach often yields reasonable results at or near specular angles.

Consider also a closed, non-penetrable conductor, whose surface is not completely PEC but slightly lossy with surface resistance R (in Ohms per square). In this case, the electric field penetrates a small distance into the conductor, and the tangential electric field on the surface does not completely vanish. As a result, the Leontovich IBC may also be used to model this type of lossy conductor, where $Z = R$.

We will now derive an IBC-EFIE and IBC-MFIE for the Leontovich IBC, using the same approach taken to derive the EFIE and MFIE for dielectric interfaces.

3.7.1.1 IBC-EFIE

First we derive the IBC-EFIE. To begin, we first note that

$$\mathbf{E} = \mathbf{E}_{tan} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{E})\,, \tag{3.215}$$

and so,

$$\mathbf{E} = Z\hat{\mathbf{n}} \times \mathbf{H} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{E}) = Z\mathbf{J} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{E})\,. \tag{3.216}$$

Taking the cross product with $\hat{\mathbf{n}}$ then yields

$$\hat{\mathbf{n}} \times \mathbf{E} = Z\hat{\mathbf{n}} \times \mathbf{J}\,. \tag{3.217}$$

Using (3.75), where in dielectric region R_l, on bounding surface S_l with normal $\hat{\mathbf{n}}_l$, the total electric field $\mathbf{E}_l(\mathbf{r})$ can be written as

$$\mathbf{E}_l(\mathbf{r}) = \mathbf{E}_l^i - j\omega\mu_l(\mathcal{L}\mathbf{J}_l)(\mathbf{r}) - (\mathcal{K}\mathbf{M}_l)(\mathbf{r}) \tag{3.218}$$

where $\mathbf{E}_l^i(\mathbf{r})$ is the incident electric field in region R_l. Taking the cross product of (3.218) with $\hat{\mathbf{n}}_l$, and using (3.217), we then obtain

$$\hat{\mathbf{n}}_l \times \mathbf{E}_l(\mathbf{r}) = Z\hat{\mathbf{n}}_l \times \mathbf{J}_l(\mathbf{r}) = \hat{\mathbf{n}}_l \times \left[\mathbf{E}_l^i(\mathbf{r}) - j\omega\mu_l(\mathcal{L}\mathbf{J}_l)(\mathbf{r}) - (\mathcal{K}\mathbf{M}_l)(\mathbf{r})\right] . \tag{3.219}$$

Taking the cross product of (3.219) with $\hat{\mathbf{n}}_l$ then yields the IBC-EFIE given by

$$\left[j\omega\mu_l(\mathcal{L}\mathbf{J}_l)(\mathbf{r}) + (\mathcal{K}\mathbf{M}_l)(\mathbf{r})\right]_{tan} + Z\mathbf{J}_l(\mathbf{r}) - \frac{1}{2}\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{M}_l(\mathbf{r}) = \left[\mathbf{E}_l^i(\mathbf{r})\right]_{tan} , \tag{3.220}$$

where the $-\frac{1}{2}\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{M}_l(\mathbf{r})$ term comes from the singularity in $(\mathcal{K}\mathbf{M}_l)(\mathbf{r})$ when $\mathbf{r} \to \mathbf{r}'$.

3.7.1.2 IBC-MFIE

Next, we consider the IBC-MFIE, which follows a dual procedure to the IBC-EFIE. To begin, we use (3.213) and (3.217) to write

$$\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H} = \hat{\mathbf{n}} \times \mathbf{J} = \frac{1}{Z}\hat{\mathbf{n}} \times \mathbf{E} = -\frac{1}{Z}\mathbf{M} , \tag{3.221}$$

or

$$\mathbf{H}_{tan} = \mathbf{H} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{H}) = \frac{1}{Z}\mathbf{M} . \tag{3.222}$$

It then follows that

$$\hat{\mathbf{n}} \times \mathbf{H} = \frac{1}{Z}\hat{\mathbf{n}} \times \mathbf{M} . \tag{3.223}$$

Then using (3.76), where in similar fashion to (3.218), the total magnetic field $\mathbf{H}_l(\mathbf{r})$ can be written as

$$\mathbf{H}_l(\mathbf{r}) = \mathbf{H}_l^i(\mathbf{r}) - j\omega\epsilon_l(\mathcal{L}\mathbf{M}_l)(\mathbf{r}) + (\mathcal{K}\mathbf{J}_l)(\mathbf{r}) \tag{3.224}$$

where $\mathbf{H}_l^i(\mathbf{r})$ is the incident magnetic field in region R_l. Taking the cross product of (3.224) with $\hat{\mathbf{n}}_l$, and using (3.223), we then obtain

$$\hat{\mathbf{n}}_l \times \mathbf{H}_l(\mathbf{r}) = \frac{1}{Z}\hat{\mathbf{n}}_l \times \mathbf{M}_l(\mathbf{r}) = \hat{\mathbf{n}}_l \times \left[\mathbf{H}_l^i(\mathbf{r}) - j\omega\epsilon_l(\mathcal{L}\mathbf{M}_l)(\mathbf{r}) + (\mathcal{K}\mathbf{J}_l)(\mathbf{r})\right] . \tag{3.225}$$

Taking the cross product of (3.225) with $\hat{\mathbf{n}}_l$, and doing some rearranging, then yields the IBC-MFIE given by

$$\left[j\omega\epsilon_l(\mathcal{L}\mathbf{M}_l)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_l)(\mathbf{r})\right]_{tan} + \frac{\mathbf{M}_l(\mathbf{r})}{Z} + \frac{1}{2}\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{J}_l(\mathbf{r}) = \left[\mathbf{H}_l^i(\mathbf{r})\right]_{tan} , \tag{3.226}$$

where the $\frac{1}{2}\hat{\mathbf{n}}_l(\mathbf{r}) \times \mathbf{J}_l(\mathbf{r})$ term comes from the singularity in $(\mathcal{K}\mathbf{J}_l)(\mathbf{r})$ when $\mathbf{r} \to \mathbf{r}'$.

3.7.1.2.1 Implementation Notes

The IBC-EFIE and IBC-MFIE are very similar to the EFIE and MFIE for dielectrics, except that now there are extra terms involving the electric and magnetic currents, and sign differences. They are combined in the same way as the EFIE and MFIE are in the PMCHWT formulation, except that they are applied only on the *exterior* of a closed surface, with no currents or fields inside.

3.7.2 Resistive Boundary

The resistive boundary, also commonly referred to as a resistive sheet, resistive card, or "r-card", comprises a very thin sheet of material of finite conductivity that is partially penetrable. When the conductivity of the sheet is very high, it is similar to a conductor with almost no penetration or power loss, and when the conductivity is very low, it is almost transparent. Thin layers of lossy dielectric, or thin conductors whose thickness is less than the skin depth, are examples of interfaces that can be modeled as a resistive sheet [18, 20]. As a practical example, thin resistive coatings are often applied to objects to reduce their radar cross section, so the ability to model these in electromagnetic field solvers is of great interest.

Consider now a resistive boundary S between two homogeneous dielectric regions R_1 and R_2, which have dielectric parameters ϵ_1, μ_1 and ϵ_2, μ_2, respectively, where the surface normal $\hat{\mathbf{n}}_1$ on S points into R_1, and $\hat{\mathbf{n}}_2$ on S points into R_2, with $\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$. The resistive boundary has a surface resistivity R measured in Ω/sq, where the tangential electric field is continuous across the sheet, but the tangential magnetic field is discontinuous. Thus, $[\mathbf{E}_1]_{tan} = [\mathbf{E}_2]_{tan}$, and the other boundary conditions on S are [18, 21]

$$\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_1 \times \left[\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})\right] = -2R\,\hat{\mathbf{n}}_1 \times \left[\mathbf{H}_1(\mathbf{r}) - \mathbf{H}_2(\mathbf{r})\right] \tag{3.227}$$

or

$$\hat{\mathbf{n}}_1 \times \left[\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})\right] = 2R\,\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_1 \times \left[\mathbf{H}_1(\mathbf{r}) - \mathbf{H}_2(\mathbf{r})\right] , \tag{3.228}$$

and

$$\hat{\mathbf{n}}_1 \times \mathbf{E}_1(\mathbf{r}) = -\hat{\mathbf{n}}_2 \times \mathbf{E}_2(\mathbf{r}) , \tag{3.229}$$

$$\hat{\mathbf{n}}_1 \times \mathbf{E}_1(\mathbf{r}) = -\mathbf{M}_1(\mathbf{r}) \tag{3.230}$$

$$\hat{\mathbf{n}}_2 \times \mathbf{E}_2(\mathbf{r}) = -\mathbf{M}_2(\mathbf{r}) \tag{3.231}$$

$$\hat{\mathbf{n}}_1 \times \mathbf{H}_1(\mathbf{r}) = \mathbf{J}_1(\mathbf{r}) , \tag{3.232}$$

$$\hat{\mathbf{n}}_2 \times \mathbf{H}_2(\mathbf{r}) = \mathbf{J}_2(\mathbf{r}) , \tag{3.233}$$

and it follows directly that

$$\mathbf{M}_1(\mathbf{r}) = -\mathbf{M}_2(\mathbf{r}) . \tag{3.234}$$

We will now derive a Resistive Boundary EFIE (RB-EFIE) and Resistive Boundary MFIE (RB-MFIE) for the resistive boundary, using the same approach taken to derive the EFIE and MFIE for dielectric interfaces.

3.7.2.0.1 RB-EFIE

First we derive the RB-EFIE. To begin, we first note that by carrying out the first cross product, and combining boundary conditions, (3.227) can be written as

$$\big[\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})\big]_{tan} = 2R\big[\mathbf{J}_1(\mathbf{r}) + \mathbf{J}_2(\mathbf{r})\big] \,, \tag{3.235}$$

and since the tangential electric fields are continuous, we obtain

$$\big[\mathbf{E}_1(\mathbf{r})\big]_{tan} = \big[\mathbf{E}_2(\mathbf{r})\big]_{tan} = R\big[\mathbf{J}_1(\mathbf{r}) + \mathbf{J}_2(\mathbf{r})\big] \,. \tag{3.236}$$

Following the previous derivation of the total tangential electric field which converts (3.219) into (3.220), we can perform a similar substitution using (3.236) which yields the RB-EFIE for a resistive boundary S in region R_1 given by

$$\big[j\omega\mu_1(\mathcal{L}\mathbf{J}_1)(\mathbf{r}) + (\mathcal{K}\mathbf{M}_1)(\mathbf{r})\big]_{tan} + R\big[\mathbf{J}_1(\mathbf{r}) + \mathbf{J}_2(\mathbf{r})\big] - \frac{1}{2}\hat{\mathbf{n}}_1(\mathbf{r}) \times \mathbf{M}_1(\mathbf{r}) = \big[\mathbf{E}_1^i(\mathbf{r})\big]_{tan} \,. \tag{3.237}$$

The RB-EFIE on the opposite side of S in region R_2 is written similarly as

$$\big[j\omega\mu_2(\mathcal{L}\mathbf{J}_2)(\mathbf{r}) + (\mathcal{K}\mathbf{M}_2)(\mathbf{r})\big]_{tan} + R\big[\mathbf{J}_1(\mathbf{r}) + \mathbf{J}_2(\mathbf{r})\big] - \frac{1}{2}\hat{\mathbf{n}}_2(\mathbf{r}) \times \mathbf{M}_2(\mathbf{r}) = \big[\mathbf{E}_2^i(\mathbf{r})\big]_{tan} \,. \tag{3.238}$$

The existence of the incident field on the right-hand side of (3.237) and (3.238) depends on the problem. For scattering problems, it is nonzero only when R_1 or R_2 is the unbounded exterior region.

3.7.2.0.2 RB-MFIE

Next we consider the RB-MFIE. To begin, we form the cross product of $\hat{\mathbf{n}}_1$ with (3.224), which yields in region R_1

$$\big[j\omega\epsilon_1(\mathcal{L}\mathbf{M}_1)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_1)(\mathbf{r})\big]_{tan} + \frac{1}{2}\hat{\mathbf{n}}_1(\mathbf{r}) \times \mathbf{J}_1(\mathbf{r}) = \big[\mathbf{H}_1^i(\mathbf{r}) - \mathbf{H}_1(\mathbf{r})\big]_{tan} \,, \tag{3.239}$$

where as before, the $\frac{1}{2}\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{J}(\mathbf{r})$ term in each comes from the singularity in $(\mathcal{K}\mathbf{J})(\mathbf{r})$ when $\mathbf{r} \to \mathbf{r}'$. We next write (3.228) as

$$\hat{\mathbf{n}}_1 \times \mathbf{E}_1(\mathbf{r}) - \hat{\mathbf{n}}_2 \times \mathbf{E}_2(\mathbf{r}) = 2R\big[\mathbf{H}_2(\mathbf{r}) - \mathbf{H}_1(\mathbf{r})\big]_{tan} \,, \tag{3.240}$$

and using the other boundary conditions, this can be written as

$$-\mathbf{M}_1(\mathbf{r}) + \mathbf{M}_2(\mathbf{r}) = 2R\big[\mathbf{H}_2(\mathbf{r}) - \mathbf{H}_1(\mathbf{r})\big]_{tan} \tag{3.241}$$

and

$$\frac{\mathbf{M}_2(\mathbf{r})}{R} = -\frac{\mathbf{M}_1(\mathbf{r})}{R} = \big[\mathbf{H}_2(\mathbf{r}) - \mathbf{H}_1(\mathbf{r})\big]_{tan} \,, \tag{3.242}$$

and finally,

$$\big[\mathbf{H}_1(\mathbf{r})\big]_{tan} = \big[\mathbf{H}_2(\mathbf{r})\big]_{tan} + \frac{\mathbf{M}_1(\mathbf{r})}{R} \,. \tag{3.243}$$

Substitution of (3.243) into (3.239) then yields

$$\left[j\omega\epsilon_1(\mathcal{L}\mathbf{M}_1)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_1)(\mathbf{r})\right]_{tan} + \frac{1}{2}\hat{\mathbf{n}}_1(\mathbf{r}) \times \mathbf{J}_1(\mathbf{r}) + \frac{\mathbf{M}_1(\mathbf{r})}{R} = \left[\mathbf{H}_1^i(\mathbf{r}) - \mathbf{H}_2(\mathbf{r})\right]_{tan}, \tag{3.244}$$

and a similar version of (3.239) can be written for region R_2, which is

$$\left[j\omega\epsilon_2(\mathcal{L}\mathbf{M}_2)(\mathbf{r}) - (\mathcal{K}\mathbf{J}_2)(\mathbf{r})\right]_{tan} + \frac{1}{2}\hat{\mathbf{n}}_2(\mathbf{r}) \times \mathbf{J}_2(\mathbf{r}) = \left[\mathbf{H}_2^i(\mathbf{r}) - \mathbf{H}_2(\mathbf{r})\right]_{tan}. \tag{3.245}$$

Subtraction of the second from the first eliminates $[\mathbf{H}_2(\mathbf{r})]_{tan}$ from the right-hand side of both equations. Equations (3.244) and (3.245) comprise the RB-MFIE for the exterior and interior side of S in R_1 and R_2, respectively. As with the RB-EFIE, the existence of an incident field on the right-hand side of each equation depends on the problem. Also note that $\mathbf{M}_1(\mathbf{r})/R$ term appears *only* on the exterior of S in R_1.

Other references often derive the RB-MFIE in the context of just two dielectric regions, first by making the substitution $\mathbf{M}_2(\mathbf{r}) = -\mathbf{M}_1(\mathbf{r})$ and $\mathbf{H}_2^i(\mathbf{r}) = 0$, and by then subtracting (3.245) from (3.244) to yield a single combined RB-MFIE. As realistic problems may have many dielectric regions, they are kept separate here.

3.7.2.0.3 Implementation Notes

As with the IBC-EFIE and IBC-MFIE, the RB-EFIE and RB-MFIE are combined using the PMCHWT formulation. We note that since the electric currents $\mathbf{J}_1(\mathbf{r})$ and $\mathbf{J}_2(\mathbf{r})$ are independent, and not of equal amplitude and opposite orientation as in the PMCHWT, a resistive boundary will have twice the number of electric basis functions than a regular dielectric boundary. Also, as the RB-EFIE and RB-MFIE on opposite sides of S are combined, we do not need to explicitly subtract (3.245) from (3.244).

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